Optimal Control of a Quadcopter: A Constrained Optimization Problem

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***Abstract*—In recent era, the quadcopters have become popular particularly due to their extension as unmanned aerial vehicles and remotely piloted aerial systems. The critical nature of its flight control is one of the main concerns for its advanced operations. These advanced operations involve on-board stabilization, trajectory tracking, unpredictable changes in the environment, and resilience to noisy data from small sensor systems. In this work, the quadcopter has been considered as a constrained problem, where optimization is required during transition between different set-points of advanced flight control, operations in action and reaction pairs. Firstly, dynamics of quadcopter along with its translational motion have been derived. Secondly, constrained optimization cost function has been set, which is the main building block for deriving the augmented Lagrangian optimization. Thirdly, co-states of the derived constraints are determined to maintain convergence. Performance evaluation showed accuracy of the proposed scheme.**

***Index Terms*—Augmented Lagrangian, estimation, flight estimation, optimal control, optimization, quadcopter, unmanned aerial vehicle.**

I. Introduction

Quadcopters belong to a class of autonomous robots which do not require any supporting surface. The structure of quadcopter involves four rotors which are symmetric to each other. The symmetry of these rotors is critical since the control of the quadcopter is dependent on: 1) speed variation of these rotors, and 2) variation of lift forces produced by them [1, 2].

This structural design of quadcopters has resulted in the increase of their role expectations in different applications. These applications include variety of aerospace areas, critical monitoring, military security systems, and mineral exploration. The bottleneck for all these applications is the sensitive nature of its flight control.

The flight control of quadcopter has been proposed by several control techniques. For example, [3] has talked about a Lyapumov-based control of an indoor micro-quadcopter. Combinations of PID-based control have been applied in [4]. A feedback linearization-based control is proposed in [5, 6]. Model predictive control has been applied with constraint variations in [7–12]. However, there is still a window for an optimal control of quadcopter, where hard and soft constraints, set-points, action and reaction pairs could operate within common boundary conditions.

In this paper, the quadcopter flight control has been considered as a constrained optimization problem. The proposed solution has considered an Augmented Lagrangian-based optimization. This will guarantee the quadcopter with an effective flight control in the presence of transitional set-points and degrees of freedom.

This paper is written as follows. The proposed formulation is developed in Section II. The implementation and evaluation of the proposed scheme is analyzed in Section III. Finally, some conclusions are drawn in Section IV.

II. Problem Formulation

The problem formulation is built on an Augmented Lagrangian-based constrained optimization framework. The details of this framework are expressed in Fig. 1.

A. Augmented Lagrangian-Based Constrained Optimization Framework

The details of this framework is explained in Fig. 1. 1) The state representation is made here, 2) An observation model is built on this state representation, dynamics of quadcopter are derived in 3) angular velocity of longitudinal motion, 4) angular velocity of lateral motion, 5) angular velocity of yaw motion, and (6)-(8) translational equations of motion. An energy-based objective function for AL-based optimization is expressed in (9). This is followed by building a cost function in (10). Constraints of the cost function are then defined. Augmented Lagrangian is derived with the cost function in (11)-(17), and finally co-state representation is made in (18)-(19).

1) State Representation: Consider a discrete-time dynamical system of a quadcopter [13] as shown in Fig. 1. It is assumed that the quadopter has a body frame, which is at the center of quadcopter body. Here the x-axis is pointing towards propeller 1, y-axis pointing towards propeller 2, and the z-axis is pointing to the ground. It has been assumed that the quadcopter has only three degrees of freedom for motion, which are 1) longitudinal motion, 2) lateral motion, and 3) yaw motion. The state representation of such a model can be expressed as:

$$\begin{aligned} x(t+1) = F(t)x(t) + \aleph(t)\varOmega_{x,\phi}(t) + \Im(t)\varOmega_{y,\theta}(t) \\ + \varpi(t)\varOmega_{z,\psi}(t) + \mathcal{G}(t)w(t) \end{aligned} \quad (1)$$

where $x_0(t) \in \mathbf{R}^{n\times 1}$ is the initial condition of the state of copter at time-instant t. $F(t) \in \mathbf{R}^{n\times n}$ is a model matrix of the state response, $\varOmega_{x,\phi}(t) \in \mathbf{R}^{n\times 1}$ is the variable of angular velocity for longitudinal motion. Subscript ϕ represents the roll rotation at x-axis. $\aleph(t) \in \mathbf{R}^{n\times n}$ is the transition matrix of longitudinal motion. $\varOmega_{y,\theta}(t) \in \mathbf{R}^{n\times 1}$ is the variable of angular velocity for lateral motion. Subscript θ represents the pitch rotation at y-axis. $\Im(t) \in \mathbf{R}^{n\times n}$ is the transition matrix of lateral motion. $\varOmega_{z,\psi}(t) \in \mathbf{R}^{n\times 1}$ is the variable of angular velocity for yaw motion. Subscript ψ represents the yaw rotation at z-axis. $\varpi(t) \in \mathbf{R}^{n\times n}$ is the transition matrix of yaw motion. $G(t) \in \mathbf{R}^{n\times n}$ is the noise transition matrix, which can be defined as a probability vector whose elements are non-negative real numbers and sum to 1. $w(t) \in \mathbf{R}^{n\times 1}$ is the random process noise, t is the time instant, and T refers to the number of time instants.

2) Observation Model: Let the multi-rotor system in (1) be observed at time-instant t as:

$$y(t) = H(t)x(t) + \nu(t) \quad (2)$$

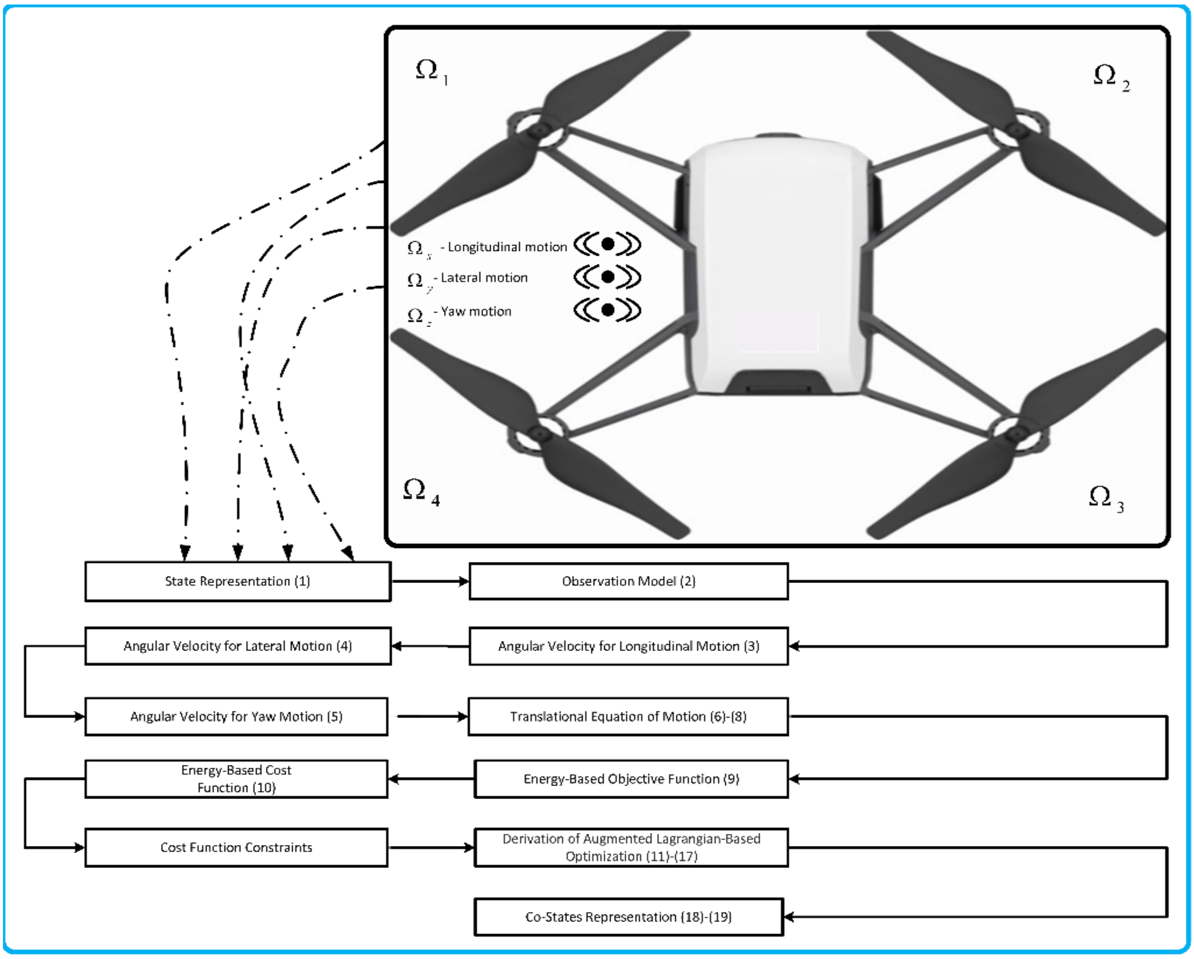

Fig. 1. Proposed framework of constrained optimization

where $y(t) \in \mathbf{R}^{m\times 1}$ is the observation output of state of multi-rotor system, m is the number of simultaneous observations for estimation made at time instant t, $H(t) \in \mathbf{R}^{m\times n}$ is the observation matrix of state, and $\nu(t) \in \mathbf{R}^{m\times 1}$ is the observation noise.

Once the observation model is extracted from the dynamics of quadcopter, the motion equations for each degree of freedom are expressed.

3) Angular Velocity for Longitudinal Motion: Quadcopter equations of motion can be expressed in detail in [13]. I_{xx}, I_{yy} and I_{zz} are the inertial area moments around each axis. The angular velocity for longitudinal motion $\Omega_{x,\phi}(t)$ can be represented in the form of roll rotation angle ϕ at x-axis as:

$$\ddot{\phi}(t) = \frac{1}{I_{xx}(t)}U_2(t) - \frac{J_r(t)}{I_{xx}(t)}\dot{\theta}(t)\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I_{xx}(t)}\dot{\psi}(t)\dot{\theta}(t) - \frac{I_{zz}(t)}{I_{xx}(t)}\dot{\theta}(t)\dot{\psi}(t) \quad (3)$$

where $U_2(t)$ is the control vector for the desired roll angle, and J_r is the inertia of rotor.

4) Angular Velocity for Lateral Motion: The angular velocity for lateral motion $\Omega_{y,\theta}(t)$ can be represented in the form of pitch rotation angle θ at y-axis as:

$$\ddot{\theta}(t) = \frac{1}{I_{yy}(t)}U_3(t) - \frac{J_r(t)}{I_{yy}(t)}\dot{\phi}(t)\Omega_{y,\theta}(t) + \frac{I_{zz}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\psi}(t) - \frac{I_{xx}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\theta}(t) \quad (4)$$

where $U_3(t)$ is the control vector for the desired pitch angle.

5) Angular Velocity for Yaw Motion: The angular velocity for yaw motion $\Omega_{z,\psi}(t)$ can be represented in the form of yaw rotation angle ψ at z-axis as:

$$\ddot{\psi}(t) = \frac{1}{I_{zz}(t)}U_4(t) + \frac{I_{xx}(t)}{I_{zz}(t)}\dot{\theta}(t)\dot{\phi}(t) - \frac{I_{yy}(t)}{I_{zz}(t)}\dot{\phi}(t)\dot{\theta}(t) \quad (5)$$

where $U_4(t)$ is the control vector for the desired yaw angle.

6) Translational Equations of Motion: The translational equation of motion in the Earth's inertial frame along x, y and z-axis can be represented as:

$$\ddot{x}(t) = \frac{-U_1(t)}{m}\Big(\sin\phi(t)\sin\psi(t) + \cos\phi(t)\cos\psi(t)\sin\theta(t)\Big) \quad (6)$$

$$\ddot{y}(t) = \frac{-U_1(t)}{m}\Big(\cos\phi(t)\sin\psi(t)\sin\phi\theta(t)$$

$$- \cos\psi(t)\sin\phi(t)\Big) \quad (7)$$

$$\ddot{z}(t) = g - \frac{U_1(t)}{m}\Big(\cos\phi(t)cos\theta(t)\Big) \quad (8)$$

where $U_1(t)$ is the control vector for the resulting upward force, m is the quadrotor's mass, and g is the gravitational acceleration.

To control the degrees of motion, the computation of individual dynamics is required. This is handled by developing a constrained optimization model for the system, which could optimize each motion variable by considering it as a subproblem. This would contribute towards the original objective function of controlling the quadcopter. In order to achieve that, an augmented Lagrangian-based solution is proposed here.

7) Energy-based Objective Function: The objective of optimal control is to bring the estimated $\ddot{\phi}(t)$, $\ddot{\theta}(t)$, $\ddot{\psi}(t)$, $\ddot{x}(t)$, $\ddot{y}(t)$ and $\ddot{z}(t)$ as close as possible to the reference (set-point) profiles. The energy-based objective function $f(E)$ is defined as:

$$\begin{aligned} f(E)=&\Big(\phi(t)-\hat{\phi}(t)\Big)^{'}\Big(\phi(t)-\hat{\phi}(t)\Big)+\Big(\theta(t)-\hat{\theta}(t)\Big)^{'} \\ &\Big(\theta(t)-\hat{\theta}(t)\Big)+\Big(\psi(t)-\hat{\psi}(t)\Big)^{'}\Big(\psi(t)-\hat{\psi}(t)\Big) \\ &+\Big(x(t)-\hat{x}(t)\Big)^{'}\Big(x(t)-\hat{x}(t)\Big)+\Big(y(t)-\hat{y}(t)\Big)^{'} \\ &\Big(y(t)-\hat{y}(t)\Big)+\Big(z(t)-\hat{z}(t)\Big)^{'}\Big(z(t)-\hat{z}(t)\Big) \\ &+\Delta\mathcal{U}(T)^{'}\mathcal{F}(T) \end{aligned} \quad (9)$$

where $\Delta\mathcal{U}(T)$ is the variable representing sequence of inputs from the current observed states at time-instant t, such that $\Delta\mathcal{U}(T) = [\Delta\mathcal{U}(t), ...\Delta\mathcal{U}(t+T)]$. A symbol ' over a variable represents the transpose operator. $\mathcal{F}(T)$ is a compatible vector.

8) Energy-Based Cost Function: The energy-based cost function can be represented as:

$$J_E=\frac{1}{2}\phi^2(t)+\frac{1}{2}\theta^2(t)+\frac{1}{2}\psi^2(t)+\frac{1}{2}x^2(t)+\frac{1}{2}y^2(t)+\frac{1}{2}z^2(t) \quad (10)$$

9) Cost Function Constraints: Constraints for this cost-function are defined. The first constraint represents the roll rotation angle ϕ as: $\phi(t+1) = \phi(t) + \Big[\frac{1}{I_{xx}(t)}U_2(t) - \frac{J_r(t)}{I_{xx}(t)}\dot{\theta}(t)\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I_{xx}(t)}\dot{\psi}(t)\dot{\theta}(t) - \frac{I_{zz}(t)}{I_{xx}(t)}\dot{\theta}(t)\dot{\psi}(t)\Big] \times \tau_s$. The second constraint is about the pitch rotation angle θ. It is represented as: $\theta(t+1) = \theta(t) + \Big[\frac{1}{I_{yy}(t)}U_3(t) - \frac{J_r(t)}{I_{yy}(t)}\dot{\phi}(t)\Omega_{y,\theta}(t) + \frac{I_{zz}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\psi}(t) - \frac{I_{xx}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] \times \tau_s$. The third constraint is on yaw rotation angle ψ as: $\psi(t+1) = \psi(t) + \Big[\frac{1}{I_{zz}(t)}U_4(t) + \frac{I_{xx}(t)}{I_{zz}(t)}\dot{\theta}(t)\dot{\phi}(t) - \frac{I_{yy}(t)}{I_{zz}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] \times \tau_s$. The fourth constraint is the translational equation of motion in the Earth's inertial frame along x-axis. It can be represented as: $x(t+1) = x(t) + \Big[\frac{-U_1(t)}{m}\Big(\sin\phi(t)\sin\psi(t) + \cos\phi(t)\cos\psi(t)\sin\theta(t)\Big)\Big] \times \tau_s$. The fifth constraint is the translational equation of motion in the Earth's inertial frame along y-axis. It can be represented as: $y(t+1) = y(t) + \Big[\frac{-U_1(t)}{m}\Big(\cos\phi(t)\sin\psi(t)\sin\phi\theta(t) - \cos\psi(t)\sin\phi(t)\Big)\Big] \times \tau_s$. The sixth constraint is the translational equation of motion in the Earth's inertial frame along z-axis. It can be represented as: $z(t+1) = \Big[g - \frac{U_1(t)}{m}\Big(\cos\phi(t)cos\theta(t)\Big)\Big] \times \tau_s$.

10) Derivation of Augmented Lagrangian-Based Optimization: The augmented Lagrangian-based Optimization can be derived here. This will be achieved while considering J_E as the main function, $\Xi(t)$ as lagrange multiplier, and $\Xi_r(t)$ as augmented Lagrange multiplier, such that $\Xi(t), \Xi_r(t) \geq 0$ as:

$$\begin{aligned} J_E =& \frac{1}{2}\phi^2(t) + \frac{1}{2}\theta^2(t) + \frac{1}{2}\psi^2(t) + \frac{1}{2}x^2(t) + \frac{1}{2}y^2(t) \\ &+ \frac{1}{2}z^2(t) - \Xi_1(t) \times \Big[\phi(t+1) - \phi(t) - \Big[\frac{1}{I_{xx}(t)}U_2(t) \\ &- \frac{J_r(t)}{I_{xx}(t)}\dot{\theta}(t)\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I_{xx}(t)}\dot{\psi}(t)\dot{\theta}(t) - \frac{I_{zz}(t)}{I_{xx}(t)}\dot{\theta}(t)\dot{\psi}(t)\Big] \\ &\times \tau_s\Big] - \Xi_{1,r}(t) \times \Big[\phi(t+1) - \phi(t) - \Big[\frac{1}{I_{xx}(t)}U_2(t) \\ &- \frac{J_r(t)}{I_{xx}(t)}\dot{\theta}(t)\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I_{xx}(t)}\dot{\psi}(t)\dot{\theta}(t) - \frac{I_{zz}(t)}{I_{xx}(t)}\dot{\theta}(t)\dot{\psi}(t)\Big] \\ &\times \tau_s\Big]^2 - \Xi_2(t) \times \Big[\theta(t+1) - \theta(t) - \Big[\frac{1}{I_{yy}(t)}U_3(t) \\ &- \frac{J_r(t)}{I_{yy}(t)}\dot{\phi}(t)\Omega_{y,\theta}(t) + \frac{I_{zz}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\psi}(t) - \frac{I_{xx}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] \\ &\times \tau_s\Big] - \Xi_{2,r}(t) \times \Big[\theta(t+1) - \theta(t) - \Big[\frac{1}{I_{yy}(t)}U_3(t) \\ &- \frac{J_r(t)}{I_{yy}(t)}\dot{\phi}(t)\Omega_{y,\theta}(t) + \frac{I_{zz}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\psi}(t) - \frac{I_{xx}(t)}{I_{yy}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] \\ &\times \tau_s\Big]^2 - \Xi_3(t) \times \Big[\psi(t+1) - \psi(t) - \Big[\frac{1}{I_{zz}(t)}U_4(t) \\ &- \frac{I_{xx}(t)}{I_{zz}(t)}\dot{\theta}(t)\dot{\phi}(t) - \frac{I_{yy}(t)}{I_{zz}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] \times \tau_s\Big] - \Xi_{3,r}(t) \\ &\times \Big[\psi(t+1) - \psi(t) - \Big[\frac{1}{I_{zz}(t)}U_4(t) - \frac{I_{xx}(t)}{I_{zz}(t)}\dot{\theta}(t)\dot{\phi}(t) \\ &- \frac{I_{yy}(t)}{I_{zz}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] \times \tau_s\Big]^2 - \Xi_4(t) \times \Big[x(t+1) - x(t) \\ &- \Big[\frac{-U_1(t)}{m}\Big(\sin\phi(t)\sin\psi(t) + \cos\phi(t)\cos\psi(t)\sin\theta(t)\Big)\Big] \\ &\times \tau_s\Big] - \Xi_{4,r}(t) \times \Big[x(t+1) - x(t) - \Big[\frac{-U_1(t)}{m}\Big(\sin\phi(t) \\ &\sin\psi(t) + \cos\phi(t)\cos\psi(t)\sin\theta(t)\Big)\Big] \times \tau_s\Big]^2 - \Xi_5(t) \\ &\times \Big[y(t+1) - y(t) - \Big[\frac{-U_1(t)}{m}\Big(\cos\phi(t)\sin\psi(t)\sin\theta(t) \\ &- \cos\psi(t)\sin\phi(t)\Big)\Big] \times \tau_s\Big] - \Xi_{5,r}(t) \times \Big[y(t+1) - y(t) \\ &- \Big[\frac{-U_1(t)}{m}\Big(\cos\phi(t)\sin\psi(t)\sin\theta(t) - \cos\psi(t) \\ &\sin\phi(t)\Big)\Big] \times \tau_s\Big]^2 - \Xi_6(t) \times \Big[z(t+1) - \Big[g - \frac{U_1(t)}{m} \\ &\Big(\cos\phi(t)cos\theta(t)\Big)\Big] \times \tau_s\Big] - \Xi_{6,r}(t) \times \Big[z(t+1) - \Big[g \\ &- \frac{U_1(t)}{m}\Big(\cos\phi(t)cos\theta(t)\Big)\Big] \times \tau_s\Big]^2 \end{aligned} \quad (11)$$

where $\Xi_1(t)$, $\Xi_2(t)$, $\Xi_3(t)$, $\Xi_4(t)$, $\Xi_5(t)$, and $\Xi_6(t)$ are the Lagrange multipliers. $\Xi_{1,r}(t)$, $\Xi_{2,r}(t)$, $\Xi_{3,r}(t)$, $\Xi_{4,r}(t)$, $\Xi_{5,r}(t)$,

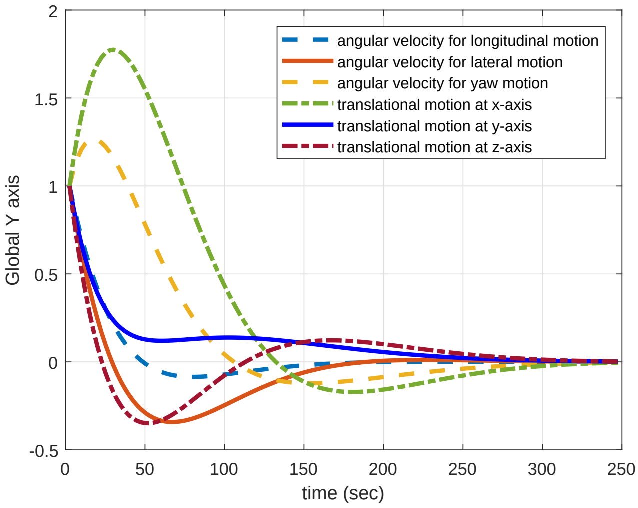

Fig. 2. Optimal control of degrees of motion using augmented Lagrangian

and $\Xi_{6,r}(t)$ are the penalty coefficients. Assuming (11) admits a 1) feasible solution, and 2) each dynamics involved when $J_E \in \zeta$ is solved, where ζ belongs to subspace $\mathbf{R}^n$.

Minimizing J_E with respect to $\phi(t)$ gives:

$$\begin{aligned}
\frac{\delta J_E}{\delta \phi(t)} &= \psi(t) + \Xi_1(t)\times\tau_s + 2\Xi_{1,r}(t)\times\psi(t)\times\tau_s \\
&- \Xi_2(t)\times\tau_s\times\Big[-\frac{J_r(t)}{I_{yy}(t)}\phi(t)\Omega_{y,\theta}(t) + \frac{I_{zz}(t)}{I_{yy}(t)}\phi(t)\dot\psi(t) \\
&- \frac{I_{xx}(t)}{I_{yy}(t)}\phi(t)\dot\theta(t)\Big] + 2\Xi_{2,r}(t)\times\tau_s\times\Big[\frac{J_r(t)}{I_{yy}(t)}\dot\phi(t) \\
&\Omega_{y,\theta}(t)\phi(t) + \frac{I_{zz}(t)}{I_{yy}(t)}\dot\phi(t)\dot\psi(t)\phi(t) - \frac{I_{xx}(t)}{I_{yy}(t)}\dot\phi(t) \\
&\dot\theta(t)\phi(t)\Big] + \Xi_3(t)\times\tau_s\times\Big[\frac{I_{xx}(t)}{I_{zz}(t)}\dot\theta(t)\phi(t) + \frac{I_{yy}(t)}{I_{zz}(t)} \\
&\phi(t)\dot\theta(t)\Big] - 2\Xi_{3,r}(t)\times\tau_s\times\Big[-\frac{I_{xx}(t)}{I_{zz}(t)}\dot\theta(t)\dot\phi(t)\phi(t) \\
&- \frac{I_{yy}(t)}{I_{zz}(t)}\dot\phi(t)\dot\theta(t)\phi(t)\Big] - \Xi_4(t)\times\tau_s\times\Big[\frac{U_1(t)}{m}\Big(cos\phi(t) \\
&\sin\psi(t) - \sin\phi(t)\cos\psi(t)\sin\theta(t)\Big)\Big] - 2\Xi_{4,r}(t)\times\tau_s \\
&\times\Big[-\Big[\frac{-U_1(t)}{m}\Big(\cos\phi(t)\sin\psi(t) - \sin\phi(t)\cos\psi(t) \\
&\sin\theta(t)\Big)\Big] - \Xi_5(t)\times\tau_s\times\Big[-\Big[\frac{U_1(t)}{m}\Big(\sin\phi(t) \\
&\sin\psi(t)\sin\theta(t) - \cos\psi(t)\cos\phi(t)\Big)\Big] - 2\Xi_{5,r}(t)\times\tau_s \\
&\times\Big[-\Big[\frac{U_1(t)}{m}\Big(\sin\phi(t)\sin\psi(t)\sin\theta(t) - \cos\psi(t) \\
&\cos\phi(t)\Big)\Big] - \Xi_6(t)\times\tau_s\times\Big[\frac{U_1(t)}{m}\Big(\sin\phi(t)cos\theta(t)\Big)\Big] \\
&- 2\Xi_{6,r}(t)\times\tau_s\times\Big[-\frac{U_1(t)}{m}\Big(\sin\phi(t)cos\theta(t)\Big)\Big] \qquad (12)
\end{aligned}$$

Minimizing J_E with respect to $\theta(t)$ gives:

$$\begin{aligned}
\frac{\delta J_E}{\delta \theta(t)} &= \psi(t) - \Xi_1(t)\times\tau_s\times\Big[-\frac{J_r(t)}{I^2_{xx}(t)}\theta(t)\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I^2_{xx}(t)} \\
&\dot\psi(t)\theta(t) - \frac{I_{zz}(t)}{I^2_{xx}(t)}\theta(t)\dot\psi(t)\Big] - 2\Xi_{1,r}(t)\times\tau_s \\
&\times\Big[-\frac{J_r(t)}{I^2_{xx}(t)}\theta(t)\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I^2_{xx}(t)}\dot\psi(t)\theta(t) - \frac{I_{zz}(t)}{I^2_{xx}(t)} \\
&\theta(t)\dot\psi(t)\Big] - \Xi_2(t)\times\tau_s\times\Big[-1 - \frac{I_{xx}(t)}{I^2_{yy}(t)}\dot\phi(t)\theta(t)\Big] \\
&- 2\Xi_{2,r}(t)\times\tau_s\times\Big[-1 - \Big[-\frac{I_{xx}(t)}{I^2_{yy}(t)}\dot\phi(t)\theta(t)\Big] \\
&- \Xi_3(t)\times\tau_s\times\Big[-\frac{I_{xx}(t)}{I^2_{zz}(t)}\theta(t)\dot\phi(t) - \frac{I_{yy}(t)}{I^2_{zz}(t)}\dot\phi(t)\theta(t)\Big] \\
&- 2\Xi_{3,r}(t)\times\tau_s\times\Big[-\frac{I_{xx}(t)}{I^2_{zz}(t)}\theta(t)\dot\phi(t) - \frac{I_{yy}(t)}{I^2_{zz}(t)} \\
&\dot\phi(t)\theta(t)\Big] - \Xi_4(t)\times\tau_s\times\Big[-\Big[\frac{-U_1(t)}{m}\Big(\cos\phi(t) \\
&\cos\psi(t)\cos\theta(t)\Big)\Big] - \Xi_{4,r}(t)\times\tau_s\times\Big[\Big[\frac{U_1(t)}{m} \\
&\Big(\cos\phi(t)\cos\psi(t)\cos\theta(t)\Big)\Big] - \Xi_5(t)\times\Big[\Big[\frac{-U_1(t)}{m} \\
&\Big(\cos\phi(t)\sin\psi(t)\cos\theta(t)\Big)\Big] - 2\Xi_{5,r}(t)\times\tau_s \\
&\times\Big[\Big[\frac{U_1(t)}{m}\Big(\cos\phi(t)\sin\psi(t)\cos\theta(t)\Big)\Big] - \Xi_6(t)\times\tau_s \\
&\times\Big[\Big[\frac{U_1(t)}{m}\Big(\cos\phi(t)sin\theta(t)\Big)\Big] - 2\Xi_{6,r}(t)\times\tau_s \\
&\times\Big[\Big[\frac{U_1(t)}{m}\Big(\cos\phi(t)\sin\theta(t)\Big)\Big] \qquad (13)
\end{aligned}$$

Minimizing J_E with respect to $\psi(t)$ gives:

$$\begin{aligned}
\frac{\delta J_E}{\delta \psi(t)} &= \psi(t) - \Xi_1(t)\times\tau_s\times\Big[\frac{I_{yy}(t)}{I^2_{xx}(t)}\psi(t)\dot\theta(t) - \frac{I_{zz}(t)}{I^2_{xx}(t)} \\
&\dot\theta(t)\psi(t)\Big] - 2\Xi_{1,r}(t)\times\tau_s\times\Big[\frac{I_{yy}(t)}{I^2_{xx}(t)}\psi(t)\dot\theta(t) \\
&- \frac{I_{zz}(t)}{I^2_{xx}(t)}\dot\theta(t)\psi(t)\Big] - \Xi_2(t)\times\tau_s\times\Big[\frac{I_{zz}(t)}{I^2_{yy}(t)}\dot\phi(t)\psi(t)\Big] \\
&- 2\Xi_{2,r}(t)\times\tau_s\times\Big[\frac{I_{zz}(t)}{I^2_{yy}(t)}\dot\phi(t)\psi(t)\Big] - \Xi_3(t)\times\tau_s \\
&+ 2\Xi_{3,r}(t)\times\tau_s - \Xi_4(t)\times\tau_s\times -\Big[\frac{-U_1(t)}{m}\Big(\sin\phi(t) \\
&\cos\psi(t) - \cos\phi(t)\sin\psi(t)\sin\theta(t)\Big)\Big] - 2\Xi_{4,r}(t)\times\tau_s \\
&\times\Big[\frac{U_1(t)}{m}\Big(\sin\phi(t)\cos\psi(t) - \cos\phi(t)\sin\psi(t)\sin\theta(t)\Big)\Big] \\
&- \Xi_5(t)\times\tau_s\times\Big[+\Big[\frac{U_1(t)}{m}\Big(\cos\phi(t)\cos\psi(t)\sin\theta(t) \\
&+ \sin\psi(t)\sin\phi(t)\Big)\Big] - 2\Xi_{5,r}(t)\times\tau_s\times -\Big[\frac{-U_1(t)}{m} \\
&\Big(\cos\phi(t)\cos\psi(t)\sin\theta(t) - \sin\psi(t)\sin\phi(t)\Big)\Big] \qquad (14)
\end{aligned}$$

Minimizing J_E with respect to $x(t)$ gives:

$$\begin{aligned}
\frac{\delta J_E}{\delta x(t)} &= x(t) - \Xi_1(t)\times\tau_s\times\Big[-\Big[\frac{1}{I^2_{xx}(t)}U_2(t) - \frac{J_r(t)}{I^2_{xx}(t)}\dot\theta(t) \\
&\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I^2_{xx}(t)}\dot\psi(t)\dot\theta(t) - \frac{I_{zz}(t)}{I^2_{xx}(t)}\dot\theta(t)\dot\psi(t)\Big] \\
&- 2\Xi_{1,r}(t)\times\tau_s\times\Big[-\Big[\frac{1}{I^2_{xx}(t)}U_2(t) - \frac{J_r(t)}{I^2_{xx}(t)}\dot\theta(t)
\end{aligned}$$

$$\Omega_{x,\phi}(t) + \frac{I_{yy}(t)}{I_{xx}^2(t)}\dot{\psi}(t)\dot{\theta}(t) - \frac{I_{zz}(t)}{I_{xx}^2(t)}\dot{\theta}(t)\dot{\psi}(t)\Big]$$
$$- \Xi_2(t) \times \tau_s \times \Big[-\Big[-\frac{1}{I_{yy}^2(t)}\dot{\phi}(t)\dot{\theta}(t) - I_{yy}I_{xx}\dot{\phi}\dot{\theta}\Big]$$
$$- \Xi_{2,r}(t) \times \tau_s \Big[\frac{1}{I_{yy}^2(t)}\dot{\phi}(t)\dot{\theta}(t) - \dot{I}_{yy}I_{xx}\dot{\phi}\dot{\theta}\Big] - 2\Xi_3(t)$$
$$\times \tau_s \times \Big[\frac{1}{I_{zz}^2(t)}\dot{\theta}(t)\dot{\phi}(t) - \dot{I}_{zz}I_{xx}\dot{\theta}\dot{\phi}\Big] - 2\Xi_{3,r}(t) \times \tau_s$$
$$\times \Big[-\frac{1}{I_{zz}^2(t)}\dot{\theta}(t)\dot{\phi}(t) - \dot{I}_{zz}I_{xx}\dot{\theta}\dot{\phi}\Big] - \Xi_4(t) \times \tau_s$$
$$- 2\Xi_{4,r}(t) \times \tau_s \tag{15}$$

Minimizing J_E with respect to $y(t)$ gives:

$$\frac{\delta J_E}{\delta y(t)} = y(t) - \Xi_1(t) \times \tau_s \times \Big[\frac{1}{I_{xx}^2(t)}\dot{\psi}(t)\dot{\theta}(t) - I_{xx}I_{yy}\psi\theta\Big]$$
$$- 2\Xi_{1,r}(t) \times \tau_s \times \Big[\frac{1}{I_{xx}^2(t)}\dot{\psi}(t)\dot{\theta}(t) - \dot{I}_{xx}I_{yy}\dot{\psi}\dot{\theta}\Big]$$
$$- \Xi_2(t) \times \tau_s \times \Big[-\frac{1}{I_{yy}^2(t)}U_3(t) - \frac{J_r(t)}{I_{yy}^2(t)}\dot{\phi}(t)\Omega_{y,\theta}(t)$$
$$+ \frac{I_{zz}(t)}{I_{yy}^2(t)}\dot{\phi}(t)\dot{\psi}(t) - \frac{I_{xx}(t)}{I_{yy}^2(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] - \Xi_{2,r}(t) \times \tau_s$$
$$\times \Big[-\Big[\frac{1}{I_{yy}^2(t)}U_3(t) - \frac{J_r(t)}{I_{yy}^2(t)}\dot{\phi}(t)\Omega_{y,\theta}(t) + \frac{I_{zz}(t)}{I_{yy}^2(t)}$$
$$\dot{\phi}(t)\dot{\psi}(t) - \frac{I_{xx}(t)}{I_{yy}^2(t)}\dot{\phi}(t)\dot{\theta}(t)\Big] - \Xi_3(t) \times \tau_s$$
$$\times \Big[-\frac{1}{I_{zz}^2(t)}\dot{\phi}(t)\dot{\theta}(t) - \dot{I}_{zz}I_{yy}\dot{\phi}\dot{\theta}\Big] - 2\Xi_{3,r}(t) \times \tau_s$$
$$\times \Big[-\frac{1}{I_{zz}^2(t)}\dot{\phi}(t)\dot{\theta}(t) - \dot{I}_{zz}I_{yy}\dot{\phi}\dot{\theta}\Big] - \Xi_5(t) \times \tau_s$$
$$- 2\Xi_{5,r}(t) \times \tau_s \tag{16}$$

Minimizing J_E with respect to $z(t)$ gives:

$$\frac{\delta J_E}{\delta z(t)} = z(t) - \Xi_1(t) \times \tau_s \times \frac{\dot{\theta}\dot{\psi}}{I_{xx}^2}\Big] - 2\Xi_{1,r}(t) \times \Big[-\frac{I_{zz}(t)}{I_{xx}(t)}$$
$$\dot{\theta}(t)\dot{\psi}(t)(\frac{-\dot{\theta}\dot{\psi}}{I_{xx}^2})\Big] - \Xi_2(t) \times \frac{\dot{\phi}\dot{\psi}}{I_{yy}^2} - 2\Xi_{2,r}(t) \times \Big[\frac{I_{zz}(t)}{I_{yy}(t)}$$
$$\dot{\phi}(t)\dot{\psi}(t) - \frac{\dot{\phi}\dot{\psi}}{I_{yy}^2}\Big] - \Xi_3(t) \times \tau_s \times \Big[-\Big[\frac{1}{I_{zz}(t)}U_4(t)$$
$$- \frac{\dot{U}_4(t)I_{zz}}{I_{zz}^2} - \frac{I_{xx}(t)}{I_{zz}(t)}\dot{\theta}(t)\dot{\phi}(t)$$
$$- \frac{I_{xx}\dot{\phi}\dot{\theta}}{I_{zz}^2} - \frac{I_{yy}(t)}{I_{zz}(t)}\dot{\phi}(t)\dot{\theta}(t) - \frac{I_{yy}\dot{\phi}\dot{\theta}}{I_{zz}^2}\Big] - 2\Xi_{3,r}(t) \times \tau_s$$
$$\times \Big[-\Big[\frac{1}{I_{zz}(t)}U_4(t) - \frac{I_{xx}(t)}{I_{zz}(t)}\dot{\theta}(t) - \frac{I_{yy}(t)}{I_{zz}(t)}\dot{\phi}(t)\dot{\theta}(t)\Big]$$
$$- \Xi_4(t) \times \tau_s \times \Big[(\frac{U_4(t)}{I_{zz}^2(t)})(\frac{I_{xx}\dot{\theta}\dot{\phi}}{I_{zz}^2})(\frac{I_{yy}\dot{\phi}\dot{\theta}}{I_{zz}^2})\Big] \tag{17}$$

11) Co-States Representation: The co-states $\Xi_k(t)$ are determined by backward integration of the adjunct state equation.

III. Numerical Results

The numerical results have been generated here. The derived degrees of freedom (3)-(5), translational equation of motion (6)-(8), energy-based objective and cost functions (9)-(10), derivation of augmented Lagrangian (11)-(17), minimization with respect to co-states (18)-(19) are considered for the simulation. The parameters considered are as follows: mass is $0.5\ kg$, $I_{xx} = I_{yy} = 4.5 \times 10^{-3} kgm^2$, $I_{zz} = 10 \times 10^{-3}\ kgm^2$. It can be seen in Fig. 2 that initially the translational motion at x and z-axis suffered to converge. This is due to their different axis of freedom. However, all the three translational motions and their angular velocities were optimally controlled while achieving the steady-state value. This is due to the property of Augmented Lagrangian-based optimization and its co-states, which ensured convergence of all degrees of motion.

IV. Conclusions

This paper considered the quadcopter as a constrained optimization problem. Several dynamics of quadcopter have been considered to achieve an adequate optimization. The cost function which was determined for the optimization considered all the inertial movements and motions. This was achieved by doing subproblem minimization of the constraints. Future work would involve to develop an uncertainty handling-based robust control to tackle all variations and perturbations of the quadcopter dynamics.

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