Stability of a Complex Traffic Flow Problem: A Delay-Dependent Perturbation Filter Design

Haris M. Khalid, *Member, IEEE*, Ishaq G. M. Al Blushi, S. Al Mualla, M. Thani, A. Khalid, R. Al Razooqi, and M. Al Shehhi

***Abstract*—Complex traffic flow is a huge concern towards achieving an efficient transportation system. It involves handling all the operational issues and effects while regulating the traffic system. These operational issues also have situation of uncertainties and vagueness. In this paper, the uncertainties in the form of perturbation have been the focus of study in the theory of traffic. To achieve that, the moving-block train model has been utilized for the perturbation analysis. Firstly, a delay-dependent moving-block model is represented. This model considered the dynamics of position and velocity in the perturbation modeling of system. These dynamics have been extracted from a measurement sensor. Secondly, an augmented Lyapunov-Krasoviskii function (ALKF) is built on this model for the realization of feasible region of stability. Numerical evaluations were made on a moving-block model. Results showed an effective implementation of the proposed scheme.**

***Keywords*—Augmented Lyapunov-Krasoviskii function (ALKF), complex networks, estimation, filter design, moving-block train, perturbation, traffic flow, traffic modeling.**

I. Introduction

Traffic flow holds a huge stake in the growth of any developing country. It is an integral component of any transportation structure. With its importance, it brings a lot of concerns and factors, which are required to be modeled and analyzed. These factors include: 1) the traffic control, and modeling, 2) transportation modeling, and planning, 3) rush hours for pedestrians, vehicles and cyclists, 4) density of vehicles on specific highways and crossings. An overlook to any of these factors may adversely hamper the traffic system with severe congestions, accidents and delays.

In railway traffic, a concept of moving-block train has been introduced [1]–[3]. This concept has superseded the other modes of railway traffic systems. This is due to the features it provides while minimizing the factors which lead to problems of complex traffic flow. On a track with multiple trains, the speed and position of each train is kept in-check with way-side control units, which hold the main key factors for the bi-directional train to ground communication [4], [5]. This computes to maintain an adequate headway with other trains on the same track. This communication holds an inherited dynamic of time-delay, which is varying to the number and position of trains on the same track. This time-delay factor can propagate perturbation while compromising on the stability of a moving-block train system [6]. Propagation of perturbation has resulted in chaotic situations particularly in power systems [7], [8], where an engineered perturbation could also be in the form of cyber-attacks [9]–[12]. In this paper, the main focus is to handle the perturbation in the moving-block train.

In the situation of moving-block system, modeling a perturbation requires to capture the smallness of errors caused by the time-delay. This also requires to consider the moving-block system as a descriptor system, which could provide access to the dynamical perturbation at any time-instant. While considering the position and velocity of each train in moving-block system with the way-side controller, an Augmented Lyapunov-Krasoviskii function (ALKF) is built to study the perturbation caused by the time-delay effects of moving-block system. This function utilizes the $L-2$ gain for providing stability to time-delay based systems [13]–[15]. The property of constructing the sufficient conditions around the time-delay has resulted in providing an adequate stability to the system.

The paper is written as follows: The proposed perturbation filter design is formulated in Section II. In Section III the implementation and evaluation of the proposed scheme is made. Conclusions are drawn in Section IV.

II. Problem Formulation

The problem is formulated based on the perturbation filter framework as shown in Fig. 1.

A. Perturbation Filter Framework

The framework is expressed in Fig. 1. It elaborates the steps involved in the framework. In the 1^{st} step, state model variable for each train is defined, which is $x_{1,t}$ for train 1, $x_{2,t}$ for train 2, and $x_{3,t}$ for train 3 respectively. Then, a general delay-dependent state model for each moving train is represented. An observation output is representing the simultaneous observations in the 2^{nd} step. A sensor model for extracting the simultaneous observation is represented in the 3^{rd} step. A velocity-based moving block train model is expressed in 4^{th} step. This model has involved all the three moving trains. This is followed by computing dynamics of perturbation, position and velocity in 5^{th} and 6^{th} step respectively. This stride will analyze the delay-dependent stability of the system in 7^{th} step. This was followed by deriving ALKF stability criteria in step 8. In the last step, a performance measure of the stability analysis is expressed.

1) Delay-Dependent State Representation: Consider a linear time-variant system with communication delay $\mathcal{C}_d$. These communication delays are considered to model a moving block-train. The general state model of the train 1 can be represented as:

$$\begin{aligned}\dot{x}_1(t) &= A_1(t)x_1(t)+\sum_{j=1}^{p} A_{1,j}(t)x_1(t-\mathcal{C}_{d,j})+B_1(t)u_1(t)\\ &+ \sum_{j=1}^{p} B_{1,j}(t)u_1(t-\mathcal{C}_{d,j})+\epsilon(t) \qquad (1)\end{aligned}$$

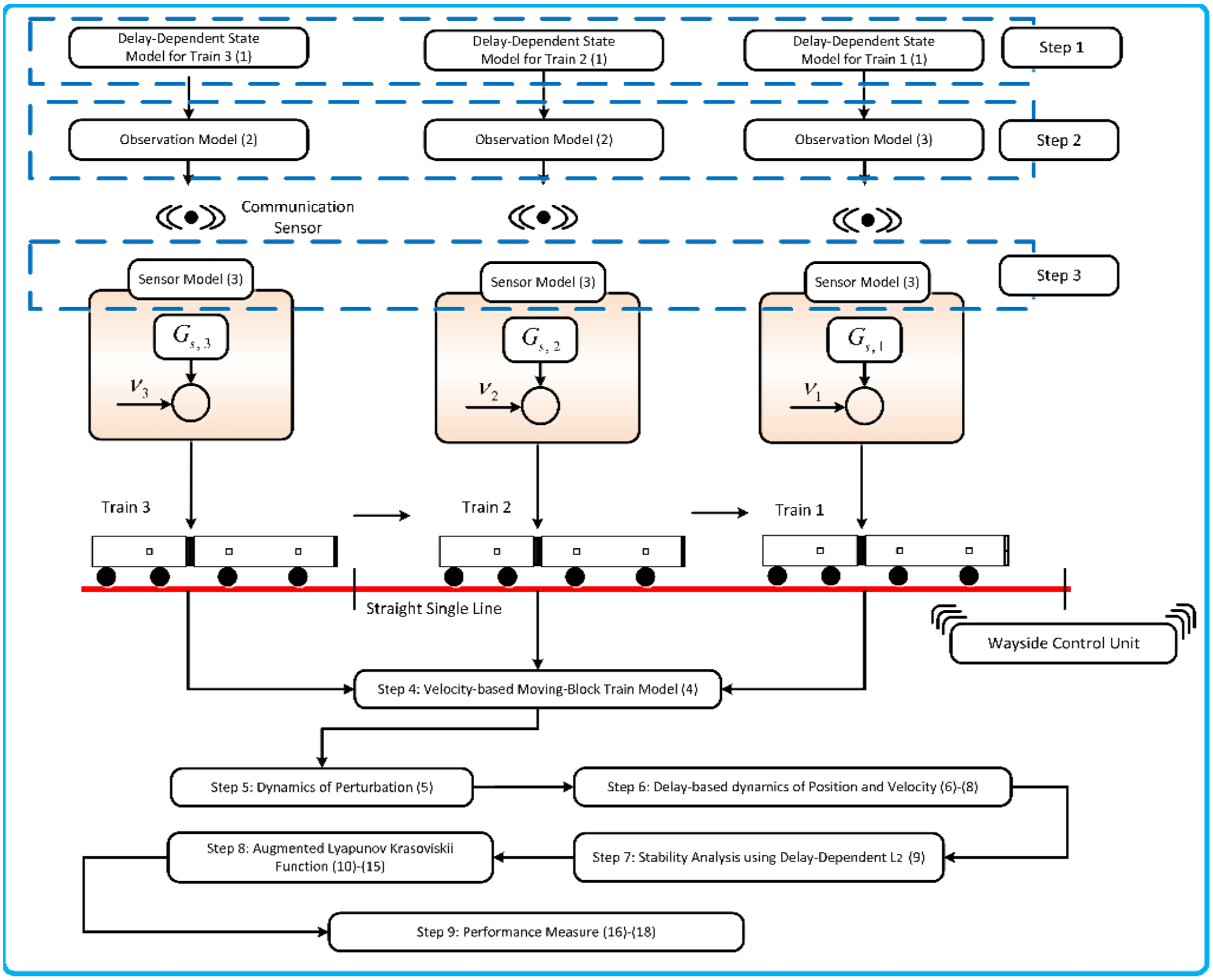

Fig. 1. Train following model for a moving-block train

where $x_1(t) \in \mathbf{R}^r$ is the state variable at time-instant t in subspace $\mathbf{R}$ for train 1. Superscript r is the size of the state vector in the subspace $\mathbf{R}$, x_0 is the initial value of state variable $x(t)$, $A(t) \in \mathbf{R}^{r\times r}$ is a model matrix of the state response, $j = 1, 2, 3, ...p$ is for all constant matrices in $\mathbf{R}^{r\times r}$, $\mathcal{C}_{d,j} = \mathcal{C}_{d,1}, \mathcal{C}_{d,2}, ...,\ \mathcal{C}_{d,p} \in \mathbf{R}^{p+}$ of which the elements are rationally independent from each other. $B(t)$ is an input transition matrix, and $u(t)$ is the input vector. $\epsilon(t)$ represents the perturbation in the moving block-train. t is the time-instant such that $t = 0, 1, 2, ...T$, where T refers to the number of time-instants.

2) Observation Model: The observation model of the moving train 1 can be stated as:

$$\Upsilon_1(t) = H_1(t)x_1(t) + H_1(t - \mathcal{C}_{d,j})x_1(t - \mathcal{C}_{d,j}) + \psi_1(t)\omega_1(t) \quad (2)$$

In the observation model (2), $\Upsilon(t) \in \mathbf{R}^{\wp}$ is the observation output of the moving block-train, $\wp$ is the number of simultaneous observations for estimation made at time-instant t, $H(t) \in \mathbf{R}^{\wp\times r}$ is the observation matrix of the state, $\psi \in \mathbf{R}^{\wp\times r}$ is a real and known constant matrix. Note the noise $w(t)$ has been assumed initially uncorrelated zero-median white Gaussian.

Once the observation model is extracted from the dynamics of electric ground vehicle, the sensor model is defined for each degree of freedom.

3) Sensor Model: A sensor is modeled by a gain and an additive noise, as given below:

$$\Upsilon_{s,k}(t) = G_{sk}\Upsilon^0_{s,k} + \nu_{s,k} \quad (3)$$

where $\Upsilon_{s,k}$, $\Upsilon^0_{s,k}$, and $\nu_{s,k}$ are the measured sensor output, true or fault-free output and additive sensor noise, respectively for a k-th sensor. Here k is equal to 1, 2 or 3 for train 1, 2, and 3 respectively. Here the gain G_{sk} is such that $0 \leq G_{s,k} \leq 1$, with the degree of the fault ranging from no fault at all for $G_{s,k} = 1$ to a complete failure for $G_{sk} = 0$.

4) Velocity-based Moving-Block Train Model: Consider a specific moving block-train system as shown in Fig. 1. It can be seen that three trains are moving on a straight line with states $x_1(t)$, $x_2(t)$ and $x_3(t)$ respectively. The lengths and locations are consistent with the location and speed of the train [6]. The moving block concept requires both knowledge of the exact location and speed of all trains at any given time. The way-side controller unit is sending command signals to trains. The linearized model can be expressed as:

$$\sum_{k=1}^{3}\dot{\chi}_k(t) = \mathcal{V}(t) + \sum_{k=1}^{3}\dot{\epsilon}_k(t) \quad (4)$$

where $\chi_k(t)$ represents the position of trains at time-instant t, $\dot{\epsilon}_k(t)$ represents the perturbations at the velocity $\mathcal{V}(t)$ of the k-th train.

5) Dynamics of Perturbation: Considering the effects of position $\mathcal{P}(t)$ and velocity $\mathcal{V}(t)$ difference between trains, the dynamics of perturbation can be represented as:

$$\begin{aligned}\ddot{\epsilon}_k(t) &= -\frac{1}{T_k(t)}\dot{\epsilon}_k(t) + S_{\mathcal{P},k}(t)[\epsilon_{k+1}(t-\mathcal{C}_{d,k+1}) - \epsilon_k(t \\ &- \mathcal{C}_{d,k})] + S_{\mathcal{V},k}(t)[\epsilon_{k+1}(t-\mathcal{C}_{d,k+1}) - \epsilon_k(t-\mathcal{C}_{d,k})] \quad (5)\end{aligned}$$

where $T_k(t)$ is the time-constant that a train changes its velocity to the velocity of the preceding train, $S_{\mathcal{P},k}(t)$ and $S_{\mathcal{V},k}(t)$ are the sensitivities of the parameters of train with respect to position and velocity errors respectively. $\mathcal{C}_{d,k}$ is the communication delay between the k-th train and the wayside controller.

6) Delay-based Dynamics of Position and Velocity: The delay differential equations based on (5) for perturbations at the position and velocity of the trains are:

$$\ddot{\epsilon}_1(t) = -\frac{1}{T_1(t)}\dot{y}_0(t) \quad (6)$$

$$\begin{aligned}\ddot{\epsilon}_2(t) &= -\frac{1}{T_2(t)}\dot{\epsilon}_2(t) + S_{\mathcal{P},2}(t)[\epsilon_1(t-\mathcal{C}_{d,2}) - \epsilon_2(t-\mathcal{C}_{d,2})] \\ &+ S_{\mathcal{V},2}(t)[\dot{\epsilon}_1(t-\mathcal{C}_{d,2}) - \dot{\epsilon}_2(t-\mathcal{C}_{d,2})] \quad (7)\end{aligned}$$

$$\begin{aligned}\ddot{\epsilon}_3(t) &= -\frac{1}{T_3(t)}\dot{\epsilon}_3(t) + S_{\mathcal{P},3}(t)[\epsilon_2(t-\mathcal{C}_{d,2}) - \epsilon_3(t-\mathcal{C}_{d,3})] \\ &+ S_{\mathcal{V},3}(t)[\dot{\epsilon}_2(t-\mathcal{C}_{d,2}) - \dot{\epsilon}_3(t-\mathcal{C}_{d,3})] \quad (8)\end{aligned}$$

Note that since the train 1 is not following another train, there is no delay term in (6) for the leading train. However, this is not the case in train 2 and 3 respectively.

Once the time delay-based dynamic relationships between the parameters of the moving block-train are determined, the perturbation involved in the system is further analyzed. This is due to the time-delay effect, which could further complicate the system analysis and may affect the system behavior and performance. To overcome these challenges, an L_2-based performance analysis is used to derive the system, which would ensure delay-dependent robust stability to the system.

7) Stability Analysis using Delay-Dependent L_2: Let ϱ and μ be the known constants to define the communication delay $\mathcal{C}_d$, such that $0 < \mathcal{C}_d(t)$, and $\dot{\mathcal{C}}_d(t) \leq \mu$. Given ϱ, $\mu > 0$, the system with $u(.) = 0$ is delay dependent asymptotically stable with L_2-performance bound γ if there exist symmetric matrices, weighting matrices and a scalar $\gamma > 0$ satisfying the following LMI:

$$\begin{bmatrix} \Upsilon_{01} & \Upsilon_{02} & \Upsilon_{03} & \varrho M_a & \varrho N_a & \mathcal{O}_p\mathcal{O}_\Gamma & \mathcal{O}_G' & \mathcal{O}_pA_0'(\mathcal{O}_a+\mathcal{O}_c) \\ * & \Upsilon_{04} & \Upsilon_{05} & \varrho M_c & \varrho N_c & 0 & \mathcal{O}_{G,d}' & \mathcal{O}_pA_d'(\mathcal{O}_a+\mathcal{O}_c) \\ * & * & \Upsilon_{06} & \varrho M_s & \varrho N_s & 0 & 0 & 0 \\ * & * & * & -\varrho\mathcal{O}_a & 0 & 0 & 0 & 0 \\ * & * & * & * & -\varrho\mathcal{O}_c & 0 & 0 & 0 \\ * & * & * & * & * & -\gamma^2 I & \Phi_0' & \mathcal{O}_p\mathcal{O}_\Gamma'(\mathcal{O}_a+\mathcal{O}_c) \\ * & * & * & * & * & * & -I & 0 \\ * & * & * & * & * & * & * & -\varrho(\mathcal{O}_a+\mathcal{O}_c) \end{bmatrix} \quad (9)$$

where $\Upsilon_{o1} = \mathcal{O}_pA_o + A_o'\mathcal{O}_p + \mathcal{O}_Q + \mathcal{O}_R + N_a + N_a' + M_a + M_a'$. Here $\mathcal{O}_p$, $\mathcal{O}_Q$, $\mathcal{O}_R$ are the symmetric matrices. N_a, N_c, N_s, M_a, M_c, and M_s are the weighting matrices. $\Upsilon_{o2} = \mathcal{O}_pA_d - 2N_a + N_c^t + M_c^t$. $\Upsilon_{o3} = N_a - M_a + N_s' + M_s'$, $\Upsilon_{o4} = -(1-\mu)\mathcal{O}_Q - 2N_c - 2N_c'$, $\Upsilon_{o5} = N_c - M_c - 2N_s'$, $\Upsilon_{o6} = -\mathcal{O}_R + N_s + N_s' - M_s - M_s'$.

However, due to the weighting norms and bounds, there is a need to enhance the feasible region of stability criteria. This is achieved by the Augmented Lyapunov-Krasoviskii function (ALKF) by utilizing additive time-delays.

8) ALKF: Consider now the ALKF as:

$$V(t) = V_o(t) + V_a(t) + V_c(t) + V_m(t) \quad (10)$$

$$V_o(t) = e_f'\mathcal{O}_Pe_f(') \quad (11)$$

$$V_a(t) = \int_{-\varrho}^{0}\int_{t+s}^{t}\dot{e}_f'(\alpha)(\mathcal{O}_a+\mathcal{O}_c)\dot{e}_f(\alpha)d\alpha ds \quad (12)$$

$$V_c(t) = \int_{t-\varrho}^{t}e_f'(s)\mathcal{O}_Re_f(s)ds \quad (13)$$

$$V_m(t) = \int_{t-\tau(t)}^{t}e_f^t(s)\mathcal{O}_Qe_f(s)ds \quad (14)$$

where $0 < \mathcal{O}_p = \mathcal{O}_p'$, $0 < \mathcal{O}_a = \mathcal{O}_a'$, $0 < \mathcal{O}_c = \mathcal{O}_c'$, $0 < \mathcal{O}_Q = \mathcal{O}_Q'$, $0 < \mathcal{O}_R = \mathcal{O}_R'$ are the matrices of appropriate dimensions. Note that the term $V_0(t)$ is representing standard to nominal system, where there is no delay involved. The terms $V_a(t)$ and $V_m(t)$ correspond to the delay-dependent conditions. The term $V_c(t)$ is introduced for a relatively large time interval. This interval can be defined from $t-\varrho \longrightarrow t$ to $t-\mathcal{C}_j \longrightarrow t$. A computation gives the time-derivative of $V(e_f)$ with $w(t) = 0$

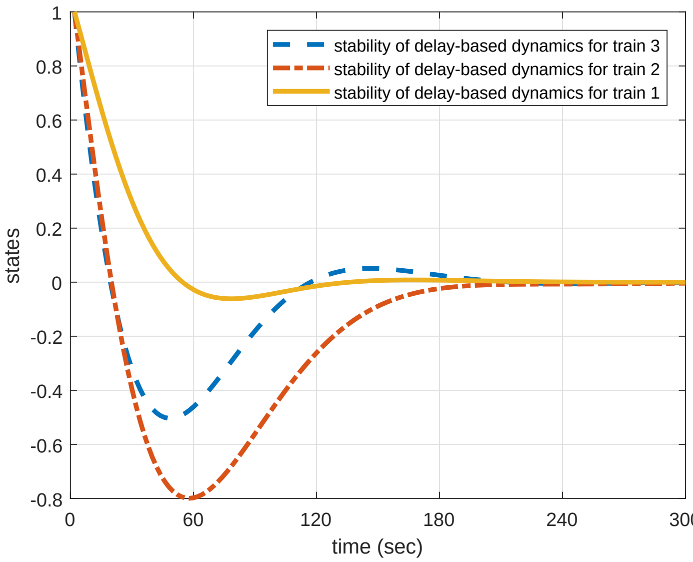

Fig. 2. Stability-of-delay-based-dynamics for train 1, 2 and 3

as:

$$\begin{aligned}&\dot{V}(t) \leq e_f^{'}(t)[\mathcal{O}_p A_o + A_o^{'} + \mathcal{O}_Q + \mathcal{O}_R + N_a + N_a^{'} + M_a \\ &+ M_a^{'}]e_f(t) + 2e_f^{'}[\mathcal{O}_p A_{do} - 2N_a + M_c^{'} + N_c^{'}]e_f(t - \mathcal{C}_j) \\ &+ 2e_f^{'}(t)[N_a - M_a + N_s^{'} + M_s^{'}]e_f(t - \varrho) + 2e_f{}^{'}(t - \mathcal{C}_j) \\ &[N_c - 2N_s^{'} - M_c]e_f(t - \varrho) - e_f{}^{'}(t - \mathcal{C}_j)[(1 - \mu)\mathcal{O}_Q \\ &+ 2N_c + 2N_c^{'}]e_f(t - \mathcal{C}_j] + e_f{}^{'}(t - \varrho)[-\mathcal{O}_R + N_s \\ &+ N_s^{'} - M_s - M_s^{'}]e_f(t - \varrho) + \varrho \dot{e_f}{}^{'}(t)(\mathcal{O}_a \\ &+ \mathcal{O}_c)\dot{e_f}(t) - \int_{t-\varrho}^{t} \dot{e_f}{}^{'}(s) \\ &(\mathcal{O}_a + \mathcal{O}_c)\dot{e_f}(s)ds - 2\xi^{'}(t)2N \int_{t-\tau(t)}^{t} \dot{e}_f(s)ds \\ &-2\xi^{'}(t)(-N) \int_{t-\varrho}^{'} \dot{e}_f(s)ds - 2\xi^{'}(t)M \int_{t-\varrho}^{t} \dot{e}_f(s)ds \quad (15)\end{aligned}$$

In view of (9) with $\mathcal{O}_G = 0$, $\mathcal{O}_{G,d} = 0$, $\mathcal{O}_\Gamma = 0$, and Schur's compliments, it follows from (15) that $\dot{V}(t) < 0$, which establishes the internal asymptotic stability.

9) Performance Measure: Consider the performance measure $J = \int_0^\infty (z^{'}(s)z(s) - \gamma^2 w^{'}(s)w(s))ds$ from the function derived in (10–15). For any $w(t) \in L_2(0, \infty) \neq 0$ and zero initial condition $x(0) = 0$, it is expressed as:

$$\begin{aligned}J &= \int_0^\infty \Big(z^{'}(s)z(s) - \gamma^2 w^{'}(s)w(s) + \dot{V}(x)\Big)ds - \dot{V}(x) \\ &\leq \int_0^\infty \Big(z^{'}(s)z(s) - \gamma^2 w^{'}(s)w(s) + \dot{V}(x)\Big)ds \qquad (16)\end{aligned}$$

This gives:

$$\begin{aligned}&z^{'}(s)z(s) - \gamma^2 w^{'}(s)w(s) + \dot{V}(s) = \bar{\mathcal{X}}^{'}(s)\bar{\Upsilon}\bar{\mathcal{X}}^{'}(s), \\ &\bar{\mathcal{X}}(s) = [e_f{}^{'}(s) \quad e_f{}^{'}(s - \mathcal{C}_j) \quad e_f{}^{'}(t - \varrho) \quad w(s)]^{'} (17)\end{aligned}$$

where $\bar{\Upsilon}$ corresponds to Υ_o in (9) by Schur's complements. It is readily seen from (9) that:

$$z^{'}(s)z(s) - \gamma^2 w^{'}(s)w(s) + \dot{V}(s) < 0 \qquad (18)$$

for arbitrary $s \in [t, \infty)$, which implies for any $w(t) \in L_2(0, \infty) \neq 0$ that $J < 0$ leading to $\| z(t) \|_2 < \gamma \| w(t) \|_2$.

III. Numerical Results

A moving-block train system with three moving trains has been considered here. Two trains are following a leading train on a straight line. The dynamics of perturbation are represented in (5), and the dynamics of position and velocity are represented in (6), (7) and (8). The numerical values of sensitivity with respect to position are: $S_{\mathcal{P},1} = S_{\mathcal{P},2} = S_{\mathcal{P},3} = 0.132$. The values of sensitivity with respect to velocity are: $S_{\mathcal{V},1} = S_{\mathcal{V},2} = S_{\mathcal{V},3} = 0.45$. It can be seen in Fig. 2 that initially the following train 2 and train 3 suffered due to time-delay and differences of position and velocity. However, all three trains eventually confirmed stability in the presence of delay. This is due to the property of proposed function which enhance the feasibility region of the delay-dependent moving trains.

IV. Conclusions

The theory of traffic is discussed in this paper from the perspective of perturbation in a moving-block train. The model considered dynamics of delay, position and velocity of the train. The implementation of the delay-dependent Lyapunov function enhanced the feasibility region of the system. Performance evaluation showed convincing results towards achieving stability. Future implications could be towards an analysis of a time-varying delay and with wireless communication in moving-block train systems.

References

[1] R. Pascoe and T. Eichorn, “What is communication-based train control?,” *IEEE Vehicular Technology Magazine*, vol. 4, no. 4, pp. 16–21, Dec. 2009.

[2] R. Bu, F. R. Yu, and T. Tang, “Performance improved methods for communication-based train control systems with random packet drops,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 15, no. 3, pp. 1179–1192, Jun. 2014.

[3] Q. Gu, T. Tang, and F. Ma,“Energy-efficient train tracking operation based on multiple optimization models,” *IEEE Transactions on Intelligent Transportation Systems*, vol. 17, no. 3, pp. 882–892, Mar. 2016.

[4] Li Zhu, F. Richard Yu, Bin Ning, Tao Tang, “Communication-based train control (CBTC) systems with cooperative relaying: Design and performance analysis,” *IEEE Transactions on Vehicular Technology*, pp. 2162–2172, vol. 63, no. 5, Jun. 2014.

[5] S. Lin, Z. Zhong, L. Cai, and Y. Luo, “Finite state Markov modelling for high speed railway wireless communication channel,” *IEEE Global Communications Conference (GLOBECOM)*, Anaheim, CA, USA, 3-7 Dec. 2012.

[6] B. Alikoc, I. Mutlu, A. F. Ergenc, “Stability analysis of train following model with multiple comunication delays,” *1st IFAC Workshop on Advances in Control and Automation Theory for Transportation Applications*, pp. 13–18, Sep. 16-17, 2013.

[7] H. M. Khalid, and J. C.-H. Peng, “Tracking electromechanical oscillations: An enhanced maximum-likelihood based approach,” *IEEE Transactions on Power Systems*, vol. 31, no. 3, pp. 1799–1808, May 2016.

[8] H. M. Khalid, and J. C.-H. Peng, “Improved recursive electromechanical oscillations monitoring scheme: A novel distributed approach,” *IEEE Transactions on Power Systems*, vol. 30, no. 2, pp. 680–688, March 2015.

[9] Ahmed S. Musleh, H. M. Khalid, S. M. Muyeen, and Ahmed Al-Durra, “A prediction algorithm to enhance grid resilience towards cyber-attacks in WAMCS applications,” *IEEE Systems Journal*, August 2017, In Press.

[10] H. M. Khalid, and J. C.-H. Peng, “Immunity towards data-injection attacks using track fusion-based model prediction,” *IEEE Transactions on Smart Grid*, vol. 8, no. 2, pp. 697–707, March 2017.

[11] H. M. Khalid, and J. C.-H. Peng, “A Bayesian algorithm to enhance the resilience of WAMS applications against cyber-attacks,” *IEEE Transactions on Smart Grid, Special Issue Theory of Complex Systems with Applications to Smart Grid Operations*, vol. 7(4), pp. 2026–2037, March 2016.

[12] A. Alamin, H. M. Khalid, and J. C. H. Peng, “Power system state estimation based on iterative extended Kalman filtering and bad data detection using normalized residual test,” *IEEE Power and Energy Conference*, Illinois, USA, 20-21 February, 2015.

[13] P. Li, J. Lam, Y. Chen, K. C. Cheung, Y. Niu, “Stability and L2-gain analysis of periodic piecewise linear systems,” *American Control Conference (ACC)*, pp. 3509–3514, Chicago, IL, USA, 1-3 July, 2015.

[14] Y. Xie, J. Wen, S. K. Nguang, L. Peng, “Stability, L2-gain, and robust H_∞ control for switched systems via N-step-ahead Lyapunov function approach,” *IEEE Access*, vol. 5, pp. 26400–26408, 2017.

[15] M. Wakaiki, K. Okano, J. P. Hespanha, “L2-gain analysis of systems with clock offsets,” *American Control Conference (ACC)*, pp. 4239–4244, Boston, MA, USA, 6-8 July, 2016.