Distributed Estimation based on Prior Information: An Improved Approximated Approach

Haris M. Khalid, *Member, IEEE* and Jimmy C.-H. Peng, *Member, IEEE* and Magdi S. Mahmoud, *Fellow, IEE*

Abstract

In this paper, a distributed estimation algorithm using Bayesian-based forward backward (FB) Kalman filter (KF) is proposed for stochastic singular linear systems. The method incorporates generalized versions of KF for bounded cases with complete and incomplete prior information, followed by estimation fusion of these cases. The incorporated filters remain optimal given the cross-covariance of the local estimates. The proposed approach is validated on a coupled tank system.

Keywords: *a-priori* information, Bayesian, distributed estimation, Kalman filtering, coupled tank system, stochastic singular linear system.

I. Introduction

ESTIMATION is one of the precise solution in providing a strict surveillance system for an appropriate supervision. One of the methods to achieve such sort of estimation often requires a group of distributed sensors which provide information of the local targets. The classic work of Rao and Durrant-Whyte [1] presents an approach to decentralized Kalman filtering which accomplishes globally optimal performance in the case where all sensors can communicate with all other sensors. Other estimation methods can be a sensor-less approach [2,3], or a derivative-free fil-

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Haris M. Khalid and Jimmy C. -H. Peng are with Department of Electrical Engineering and Computer Science, Institute Center for Energy, Masdar Institute of Science and Technology (MIST), Abu Dhabi, UAE. E-mail: mkhalid, jpeng@masdar.ac.ae
Magdi S. Mahmoud is with the Systems Engineering Department, King Fahd University of Petroleum and Minerals (KFUPM), Dhahran 31261, KSA. E-mail: msmahmoud@kfupm.edu.sa

tering estimation [4], a least-squares-Kalman technique [5], a robot-based autonomous estimation and detection [6], H_∞ filtering-based estimation made for stochastic incomplete measurements [7], sequential Bayesian learning based dual estimation method [8], process noise identification-based particle filter estimation [9], a non-linear operator based estimation [10, 11], quantized measurements-based state estimation [12], FB-KF based estimation in fault diagnosis scheme [13], state estimation for static networks using weighted filters [14] etc.

One of the methods to achieve such sort of estimation often requires a group of distributed sensors which provide information of the local targets. This kind of multi-target tracking architecture can be applied to large flexible and smart structures such as condition and health monitoring of aircrafts, industrial plants, and electrical infrastructures [15]. The problem of such an architecture utilizing information from multiple sensors employed has been in focus since last many years [16, 17]. While achieving this approach, many fusion algorithms and filters were derived to combine local estimates local estimates [18–20] to prove better efficiency and effectiveness. An obvious drawback to the multiple sensor-based information approaches is that the errors generated from observations of different sensors have to be uncorrelated. This is due to the reason that the errors generated by the local sensors are dependent upon a common information to be estimated. Also, the dynamical model of the local sensors have to be identical [21]. It has been realized for many years following the original work of [16, 22] that the local estimates have correlated errors. These errors become complicated when the prior information given to the local sensors have missing or incomplete information.

In this paper, an approximate distributed estimation is derived for different prior cases [23] with the help of Bayesian-based FB KF. The estimation is formulated based on a stochastic singular linear system. To reduce the time complexity, upper bound (ub) and lower bound (lb) methods are developed on the cases of prior knowledge for time complexity reduction. After estimation, a data fusion technique is used to encapsulate the system in a distributed structure. The proposed scheme is then validated on a bench-marked laboratory scaled coupled tank system, where leakage fault is introduced along with different profile data for the evaluation of the proposed scheme. An overview of the scheme validated on the coupled tank system is illustrated in Fig. 1. The proposed recursive scheme can adaptively estimate the parameters when the measurements are subjected to leakage fault and loss in prior information in the system. This is achieved by feeding signal estimations from each metering location. The error covariance matrix $P_{k|k}$ and state estimate $\widehat{\alpha}_{k|k}$ from each state are used as a feedback to all metering locations with a package containing the updated covariance matrix and state estimate values. This provides a novel and convenient way to enhance the modal estimations at locations that are isolated by noise and system perturbations. The updated covariance and state estimate are then computed using a distributed filtering architecture, followed by controller gain K to control the performance profile of the system.

The rest of this paper is written as follows. Problem formulation is described in Section II. The Bayesian-based FB KF with complete prior information is derived and discussed in Section III, followed by derivation of Bayesian-based FB KF with incomplete prior information in Section IV. Evaluation and testing are outlined in Section V. Finally conclusions are drawn in Section VI.

Notations: In this paper, a *widehat* over a variable indicates an estimate of the variable e.g. $\widehat{\alpha}$ is an estimate of α. The $\sim$ over a variable indicates an optimal value of the variable e.g. $\tilde{K}$ is an optimal gain of K. The *overline* over a variable indicates the upper-bound value of the variable e.g. $\overline{P}_{k|k}$ is the upper-bound of $P_{k|k}$. The *underline* under a variable indicates the lower-bound value of the variable e.g. $\underline{\alpha}_{k|k}$ is the lower-bound of $\alpha_{k|k}$. When any of these variables becomes a function of time, the time index k appears as a subscript (e.g. we write α_k, C_k, Υ_k).

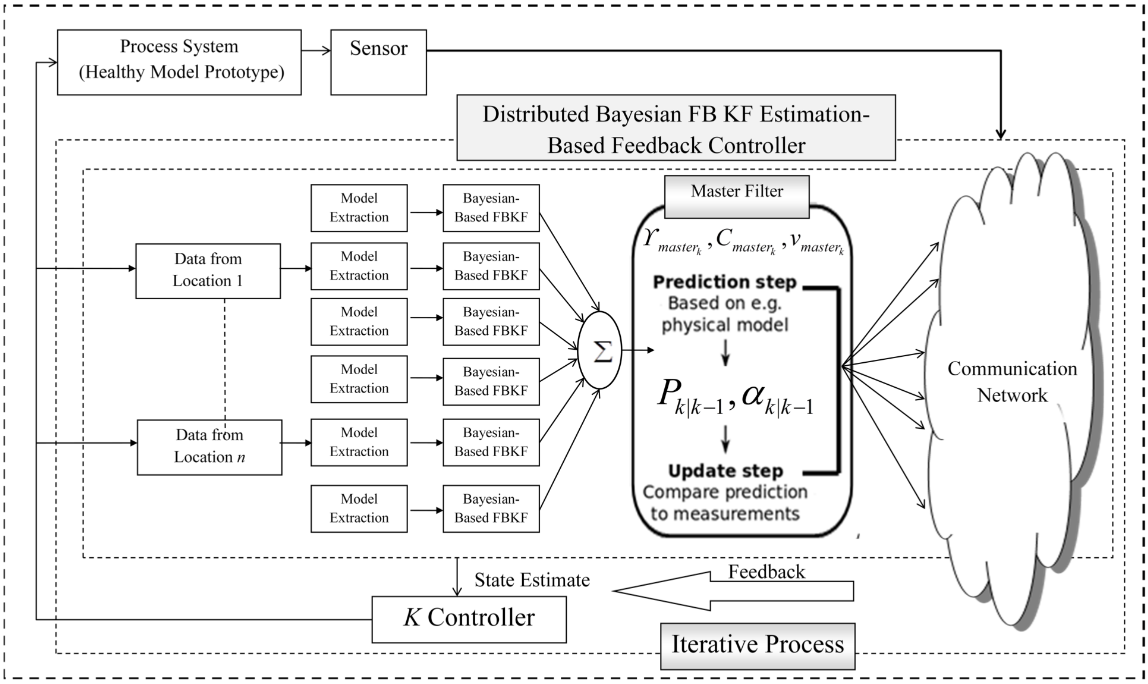

Fig. 1. Proposed distributed estimation scheme

II. Problem Formulation

Consider a stochastic singular linear system with multiple sensors representing a coupled tank system. The aim is to estimate fault α_k using the measurement equation, given by the following discrete-time model for an i-th sensor at time instant k:

$$M\alpha_{k+1} = \Phi\alpha_k + \Gamma\omega_k,\ k = 0,1\ ...,\ K \tag{1}$$

$$\Upsilon_k^i = C_k^i\alpha_k + \nu_k^i,\ i = 1,2,\ ...,\ l \tag{2}$$

where the state $\alpha_k \in \mathbf{R}^n$ represent the state (leakage profile, a type of fault), $\Phi \in \mathbf{R}^{n\times r}$ is a modal matrix of the response, such that it depends on covariates, $\omega_k^i \in \mathbf{R}^n$ is the local stochastic process with zero-mean white Gaussian noise at i-th sensor, such that $\mathbf{E}[w_k] = 0$, where $i = 1, 2, ..., l$, l is the number of sensors. M and Γ are the constant matrices with compatible dimensions, $\Upsilon_k^i \in \mathbf{R}^{m_i}$, represent the local measurement observation output of the i-th sensor e.g. the hydraulic height profile, $\nu_k^i \in \mathbf{R}^{m_i}$, represent the local measurement noise, superscript m_i is the number of local simultaneous observations made by i-th sensor at time instant k, $C_k^i \in \mathbf{R}^{m_i \times r}$ represents the local observation matrix perturbed by the fault which is to be estimated. It is assumed that ω_k, ν_k are zero mean mutually uncorrelated white noises with $\mathbf{E}\ [\omega_k^i\ \omega_k^{i^T}] = Q_w \delta_k^i$ and $\mathbf{E}\ [\nu_k^i\ \nu_k^{i^T}] = Q_\nu \delta_k^i$, where $\mathbf{E}$ denotes the mathematical expectation, Q_ω and Q_ν are constant symmetric positive semi-definite matrices, δ_k^i is the Kronecker delta used for shifting integer variable for the presence or absence of noise accordingly. The superscript T stands for the transpose. Observations from all l number of sensors in the network are integrated synthetically to the master observation output model $\Upsilon_{\text{master},k} \in \mathbf{R}^{p_{\text{master}}}$, subscript *master*, k represents the global observations gathered from the local i-th sensors at time instant k, superscript p_{master} is the master observation output collected from number of local i-th sensors. Suppose the master observation matrix, $C_{\text{master}_k} \in \mathbf{R}^{p_{\text{master}} \times r}$ and the master observation noise vector, $\upsilon_{\text{master},k} \in \mathbf{R}^{p_{\text{master}}}$ be:

$$\Upsilon_{\text{master}_k} = \begin{bmatrix} \Upsilon_k^1 \\ \vdots \\ \Upsilon_k^l \end{bmatrix}, C_{\text{master}_k} = \begin{bmatrix} C_k^1 \\ \vdots \\ C_k^l \end{bmatrix}, \upsilon_{\text{master}_k} = \begin{bmatrix} \upsilon_k^1 \\ \vdots \\ \upsilon_k^l \end{bmatrix} \tag{3}$$

where l is the number of sensors. Then the master observation model at k instance is given by:

$$\Upsilon_{\text{master},k} = C_{\text{master},k} \alpha_k + \upsilon_{\text{master},k}, \tag{4}$$

where $\Upsilon_{\text{master},k}$ is the master observation output vector, $C_{\text{master},k}$ is the master observation matrix, α_k is the state matrix, and $\upsilon_{\text{master},k}$ is the master observation noise vector. From equation (1) and (2), it is assumed that the pair $(\phi, C_{\text{master},k})$ is observable. In this paper, the following assumptions are made.

Assumption II.1: M is a singular square matrix where, $rank\, M = n_1 < n$, $rank\, \Phi \geq n_2$ and $n_1 + n_2 = n$. The system (1–2) is observable, i.e.,

$$rank \begin{bmatrix} zM - \Phi \\ C_k \end{bmatrix} = n, \forall z \in \mathcal{C};\ rank \begin{bmatrix} M \\ C_k \end{bmatrix} = n \tag{5}$$

where $\mathcal{C}$ is the set of complex numbers.

Assumption II.2: System (1–2) is regular, i.e., $det(zM - \Phi) \neq 0$ where z is an arbitrary complex. It should be noted that the estimation problem is considered under the assumption of regularity [$det(zM - \Phi) \neq 0$] and causality where matrices M and Φ are square and singular.

By letting $\Theta = inv(M)\Phi$, and $Gw_k = inv(M)\Gamma\omega_k$, a time-varying linear dynamic model can be seen as:

$$\alpha_{k+1} = \Theta\alpha_k + Gw_k \tag{6}$$

Considering a distributed networked control system, in which agents communicate with each other over a wired communication channel. Let $Z_k^{ij} \in \{0, 1\}$ be a Bernoulli random variable, such that $Z_k^{ij} = 1$ if a packet sent by the agent i is correctly received by the agent j at time k, otherwise Z_k^{ij} $= 0$. Since, there is no communication loss within an agent, $Z_k^{ii} = 1$ for all i and k. The dynamic model can then be expressed as:

$$\alpha_{k+1} = \sum_{i=1}^{N} Z_k^{ij}\Theta\alpha_k + G\omega_k \tag{7}$$

By letting $\Theta_k = (\sum_{i=1}^{N} Z_k^{ij}\bar{\Theta})\alpha_k + G\omega_k$, it can be seen that (7) is a time-varying linear dynamic model:

$$\alpha_{k+1} = \Theta_k\alpha_k + G\omega_k \tag{8}$$

Considering a more general case, where the matrix Θ_k is time-varying and its values are determined by Z_k. Note Z is a random variable for the non-singular term Φ/M. Hence, Θ is a function of Z_k and this general case can be described as:

$$\alpha_{k+1} = \Theta(Z_k)\alpha_k + Gw_k \tag{9}$$

In the following sections, the derivation of KF fusion algorithm with cases of prior information is presented [24] for an i-th sensor. The Bayesian-based FB KF is expressed as (See Eq. (10–19)), where the simple Bayesian-based optimal KF is expressed in [25]. It should be noted that the derivation of $a-priori$ knowledge proof is inspired by [26], where the estimation fusion is considered for the BLUE filters only.

$$Forward\,Run\colon For\;(k=0;\;k<K;\;+k)$$

$$R^i_{e,k} = R^i_k + C^i_k P^i_{k|k-1} C^{i^T}_k \tag{10}$$

$$\widehat{K}_k = F_k \widehat{P}^i_{k|k-1} C^{i^T}_k (C^i_k \widehat{P}^i_{k|k-1} C^{i^T}_k + R^{i^{-1}}_{e,k}) \tag{11}$$

$$e^i_k = (\Upsilon^i_k - C^i_k \widehat{\alpha}^i_{k|k-1}) \tag{12}$$

$$\widehat{\alpha}^{i^{MAP}}_{k|k} = \widehat{\alpha}^i_{k|k-1} + \widehat{K}_k e^i_k \tag{13}$$

$$\widehat{\alpha}^i_{k+1|k} = \Phi_k \widehat{\alpha}^i_{k|k} \tag{14}$$

$$\widehat{P}^i_{k+1|k} = \Phi_k P^i_{k|k-1} \Phi^T_k + G_k Q_k G^T_k - \widehat{K}_k R^i_{e,k} \widehat{K}^T_k \tag{15}$$

$$\widehat{P}^i_{k|k} = \widehat{P}^i_{k|k-1} - \Phi_k \widehat{K}_k C^i_k \widehat{P}^i_{k|k-1} \tag{16}$$

$$Backward\,Run\colon For\;(k=K-1; t\geq 0; -k)$$

$$\widehat{J}_{k-1|K} = \widehat{P}^i_{k-1|K} \Phi^T_k \widehat{P}^{i^{-1}}_{k-1|K} \tag{17}$$

$$\widehat{\alpha}^i_{k-1|K} = \widehat{\alpha}^i_{k-1|k-1} + \widehat{J}_{k-1}(\widehat{\alpha}^i_{k-1|K} - \widehat{\alpha}^i_{k-1|k}) \tag{18}$$

$$\widehat{P}^i_{k-1|K} = \widehat{P}^i_{k-1|k-1} + \widehat{J}_{k-1}(\widehat{J}_{k-1|K} - \widehat{P}^i_{k-1|k} J^T_{k-1} \tag{19}$$

For an i-th sensor, $R^i_{e,k}$ is the local covariance matrix of estimation error e_k, $P^i_{k+1|k}$ is the local predicted *a-priori* estimate covariance matrix, $\widehat{\alpha}^i_{k|k}$ is the local updated $a-posteriori$ state estimate, $\widehat{\alpha}^i_{k|k-1}$ is the local predicted $a-priori$ state estimate, and $P^i_{k|k}$ is the local updated $a-posteriori$ estimate covariance, K_k is the system gain and F_k is the state-transition model for each time-step k. Q_k is the process noise correlation factor such that $Q_k = \mathbf{E}[w_k w^T_k] = \frac{1}{\sigma^2_n}\Gamma_k \Gamma^T_k$. σ^2_n is the noise variance, Γ_k is the squared matrix for state response. Φ_k is the state transition model applied to previous state α_{k-1}. The desired estimate is $\widehat{\alpha}^i_{k|K}$ which estimates the state at k instants of time, such that time sequence K is known.

It should be noted that a smoother is employed here to reduce the noise effect. The smoother allowed more accurate estimation of various prior information versions. This is due to its nature of choosing the most refined covariance error matrix $P^i_{k|k-1}$ for an i-th sensor from the K-th iteration in the forward run. Subsequently, that K-instant is considered as the first iteration in the backward run. Note that it is up to the designers to use smoothing equations. For example, Kalman smoother for online analysis will give estimates only after the end of the experiment. This may not be acceptable. In the contrary, getting the estimates after the experiment may not matter for off-line applications.

III. Bayesian-based FB KF Fusion with Complete Prior Information

In this section, a generalized version of KF is presented with complete prior information. Complete prior information means both the prior mean and the prior covariance of the estimate are known. Consider a generalized distributed networked control system (DNCS) dynamic model (9), where w_k is a Gaussian noise with zero mean, and the measurement model (20) where $\Upsilon_k \in \mathbf{R}^{n_y}$. $C_k \in \mathbf{R}^{n_y \times N_{n_x}}$ and ν_k is a Gaussian noise with zero mean and covariance Q_k.

$$\Upsilon_k = C_k \alpha_k + \nu_k \tag{20}$$

The following theorem III.1 presents the Bayesian-based FB KF with complete prior information:

Theorem III.1:

$$Forward\,Run: For\;(k=0;\;k<K;\;+k)$$

$$\widehat{\alpha}^i_{k|k} = \Phi_k \bar{\alpha}^i_k + K_{p,k}[\Upsilon^i_k - C^i_k \alpha^i_{k+1|k} - \nu_k] \tag{21}$$

$$\widehat{\alpha}^i_{k+1|k} = \Phi_k \widehat{\alpha}^i_{k+1|k} + K_k \nu_k \tag{22}$$

$$\widehat{R}^i_{e,k} = R^i_k + C^i_k P^i_{k+1|k} C^{i^T}_k + C^i_k C^i_{xv} + (C^i_k C^i_{xv})^T \tag{23}$$

$$K_k = (\Phi_k P^i_{k+1|k} C^{i^T}_k + G_k S_k)(C^i_k P^i_{k|k} C^{i^T}_k + R^i_{e,k})^{-1} \tag{24}$$

$$\widehat{P}^i_{k+1|k} = \Phi_k P^i_{k+1|k} \Phi_k^T + G_k Q_k G_k^T - \Phi_{k+1|k} K_k R^i_{e,k} K_k^T \tag{25}$$

$$\widehat{P}^i_{k|k} = \Phi_k P^i_{k+1|k} \Phi_k^T - K_k C^i_k P^i_{k+1|k} \tag{26}$$

$$Backward\, Run\text{: } For\ (k = 0;\, k < K;\, +k)$$

$$\widehat{J}_{k-1|K} = \widehat{P}^i_{k-1|K} \Phi_k^T \widehat{P}^{i-1}_{k-1|K} \tag{27}$$

$$\widehat{\alpha}^i_{k-1|K} = \widehat{\alpha}^i_{k-1|k-1} + \widehat{J}_{k-1}(\widehat{\alpha}^i_{k-1|K} - \widehat{\alpha}^i_{k-1|k}) \tag{28}$$

$$\widehat{P}^i_{k-1|K} = \widehat{P}^i_{k-1|k-1} + \widehat{J}_{k-1}(\widehat{J}_{k-1|K} - \widehat{P}^i_{k-1|k}) J^T_{k-1} \tag{29}$$

where Eq. (21–29) represents the Bayesian-based FB KF with complete prior information. Also S^i_k is the covariance of $\tilde{\Upsilon}^i_k$ for an i-th sensor. The error covariance and the gain matrices have the following alternative forms (See Eq. (30–31)):

$$\begin{aligned} P_k &= (I - K_k C^i_k) P^i_{k+1|k+1} (I - K_k C^i_k)^T + K_k R^i_{e,k} K_k^T - (I - K_k C^i_k) G_k S_k K_k^T \\ &\quad - ((I - K_k C^i_k) G_k S_k K_k^T)^T \end{aligned} \tag{30}$$

$$K_k = (\Phi_k P^i_{k+1|k} C^{i^T}_k + G_k S_k)(R^i_{e,k} + C^i_k P^i_{k|k} G_k S_k)^{-1} \tag{31}$$

where B_k is the control-input model.

Proof: This is proved in the Appendix. ■

A. *Modified Filter with Complete Prior Information*

Based on general DNCS dynamic model (9), where Z_k is independent from Z_t for $t \neq k$, an optimal linear filter is derived. The following terms are defined to describe the modified Bayesian-based FB KF.

$$\widehat{\alpha}^i_{k|k} = \mathbf{E}[\alpha^i_k | \Upsilon^i_k]$$

$$P^i_{k|k} = \mathbf{E}[e_k e_k^T | \Upsilon^i_k]$$

$$\widehat{\alpha}^i_{k+1|k} = \mathbf{E}[\alpha^i_{k+1} | \Upsilon^i_k]$$

$$
\begin{aligned}
P^i_{k+1|k} &= \mathbf{E}[e^i_{k+1|k} e^{i^T}_{k+1|k} | \Upsilon^i_k] \\
J_{k-1|T} &= \mathbf{E}[J_{k-1|T} | P^i_{k|k}] \\
\widehat{\alpha}^i_{k-1|T} &= \mathbf{E}[e_{k-1|T} | \Upsilon^i_k] \\
P^i_{k-1|T} &= \mathbf{E}[e_{k-1|T} e^T_{k-1|T} | \Upsilon^i_k]
\end{aligned} \tag{32}
$$

where $\Upsilon^i_k = \{\Upsilon^i_t : 0 \le t \le k\}$, $e_{k|k} = \alpha^i_k - \widehat{\alpha}^i_{k|k}$, and $e^i_{k+1|k} = \alpha^i_{k+1} - \widehat{\alpha}^i_{k+1|k}$.

Suppose there are estimates $\widehat{\alpha}^i_{k|k}$ and $P^i_{k|k}$ from time k. At time $k+1$, a new measurement Υ^i_{k+1} is received and the goal is to estimate $\widehat{\alpha}^i_{k+1|k+1}$ and $P^i_{k+1|k+1}$ from $\widehat{\alpha}^i_{k|k}$, $P^i_{k|k}$ and Υ^i_{k+1}. First, $\widehat{\alpha}^i_{k+1|k}$ and $P^i_{k+1|k}$ are computed as:

$$
\begin{aligned}
\widehat{\alpha}^i_{k+1|k} &= \mathbf{E}[\alpha^i_{k+1} | \Upsilon^i_k] \\
&= \mathbf{E}[A_Z \alpha^i_k + G_k \omega^i_k | \Upsilon^i_k] \\
&= \widehat{A}_z \widehat{\alpha}^i_{k|k}
\end{aligned} \tag{33}
$$

where $\widehat{A}_z = \sum_{z \in Z} p_z A^i_z$ is the expected value of A_Z. Here $p_z = P(Z = z)$, and Z is a set of all possible communication link configurations.

The prediction covariance can be computed for an i-th sensor as:

$$
\begin{aligned}
P^i_{k+1|k} &= \mathbf{E}[e^i_{k+1|k} e^{i^T}_{k+1|k} | \Upsilon^i_k] \\
&= G_k Q_k G^T_k + \sum_{z \in Z} p_z A_z P^i_{k|k} A^T_z \\
&- K_k R^i_{e,k} K^T_k + \sum_{z \in Z} p_z A_z \widehat{\alpha}^i_{k|k} \widehat{\alpha}^{i^T}_{k|k} (A_z - \widehat{A}_Z)^T
\end{aligned} \tag{34}
$$

Given $\widehat{\alpha}^i_{k+1|k}$ and $P^i_{k+1|k}$, $\widehat{\alpha}^i_{k+1|k+1}$ and $P^i_{k+1|k+1}$ are computed as in the standard KF:

$$
\widehat{\alpha}^i_{k+1|k+1} = \Phi_k \widehat{\alpha}^i_{k+1|k} + K_{k+1}(\Upsilon_{k+1} - C^i_k \widehat{\alpha}^i_{k+1|k}) - \nu^i_k \tag{35}
$$

$$
P^i_{k+1|k+1} = \Phi_k P^i_{k+1|k} \Phi^T_k - \Phi_{k|k-1} K_{k+1} C^i_k P^i_{k+1|k} \tag{36}
$$

where $K_{k+1} = (\Phi P^i_{k+1|k} C^{i^T}_k + G_k S_k)(C^i_k P^i_{k|k} C^{i^T}_k + R^i_{e,k})^{-1}$.

B. Approximating the Filter for Complete Prior Information

The modified KF proposed in Section III.A for the general DNCS is an optimal linear filter, but the time complexity of the algorithm can be exponential in N since the size of Z is $O(2^{N(N-1)})$ in the worst case, i.e., when all agents communicate with each other. In this section, two approximate KF methods are described for the general DNCS dynamic model (6), which are more computationally efficient than the modified KF by avoiding the enumeration over Z. For an i-th sensor, since the computation of $P^i_{k+1|k}$ is the only time-consuming process, two filtering methods are proposed which can bound $P^i_{k+1|k}$. The notation $A_z \geq 0$ is used if A_z is a positive semi-definite matrix and $A_z > 0$ if A_z is a positive definite matrix.

1) lb-KF: Complete Prior Information Case : The lower-bound KF (lb-KF) for an i-th sensor is the same as the modified KF described in Section III.A, except $P^i_{k+1|k}$ is approximated by $\underline{P}^i_{k+1|k}$ and $P^i_{k|k}$ by $\underline{P}^i_{k|k}$. The covariances are updated as (See Eq. (37–38)):

$$\underline{P}^i_{k+1|k} = \widehat{A}_z \underline{P}^i_{k|k} \widehat{A}_z^T + G_k Q_k G_k^T - \underline{K}_{p,k} \underline{R}^i_{e,k} \underline{K}_{p,k} \tag{37}$$

$$\underline{P}^i_{k+1|k+1} = \Phi_k \underline{P}^i_{k+1|k} - \Phi_{k|k-1} \underline{K}_{k+1} C^i_k \underline{P}^i_{k+1|k} \tag{38}$$

where $\widehat{A}_z$ is the expected value of A_z and $\underline{K}_{k+1} = \Phi_{k+1|k} \underline{P}^i_{k+1|k} C^{i^T}_k \; (C^i_k \underline{P}^i_{k+1|k} C^{i^T}_k + R^i_{e,k})^{-1}$. Notice that $\widehat{A}_z$ can be computed in advance and the lb-KF avoids the enumeration over Z.

Lemma III.1: If $\underline{P}^i_{k|k} \leq P^i_{k|k}$, then $\underline{P}^i_{k+1|k} \leq P^i_{k+1|k}$.

Proof: This is proved in the Appendix. ■

Lemma III.2: If $\underline{P}^i_{k+1|k} \leq P^i_{k+1|k}$, then $\underline{P}^i_{k+1|k+1} \leq P^i_{k+1|k+1}$.

Proof: This is proved in the Appendix ■

Remark III.1: Finally, using Lemma III.1, Lemma III.2, and the induction hypothesis, the following theorem showing that the lb-KF maintains the state error covariance, which is upper-bounded by the state error covariance of the modified KF can be obtained.

Theorem III.2: If the lb-KF starts with an initial covariance $\underline{P}^i_{0|0}$, such that $\underline{P}^i_{0|0} \leq P^i_{0|0}$, then $\underline{P}^i_{k|k} \leq P^i_{k|k}$ for all $k \geq 0$.

2) ub-KF: Complete Prior Information Case : Similar to the lb-KF, the upper-bound KF (ub-KF) approximates $P^i_{k+1|k}$ by $\overline{P}^i_{k+1|k}$ and $P^i_{k|k}$ by $\overline{P}^i_{k|k}$. Let $\lambda_{max} = \lambda_{max}(\overline{P}^i_{k|k}) + \lambda_{max}(\widehat{\alpha}^i_{k|k}\widehat{\alpha}^{i^T}_{k|k})$, where $\lambda_{max}(S)$ denotes the maximum eigenvalue of S. The covariances are updated as following (See Eqn. (39–40)):

$$\overline{P}^i_{k+1|k} = \lambda_{max}\mathbf{E}[A_z A_z^T] - \overline{K}_p \overline{R}^i_{e,k} \overline{K}_p^T - \widehat{A}_z \overline{\alpha}^i_{k|k} \overline{\alpha}^{i^T}_{k|k} \widehat{A}_z^T + G_k Q_k G_k^T \quad (39)$$

$$\overline{P}^i_{k+1|k+1} = \Phi \overline{P}^i_{k+1|k} - \Phi \overline{K}_{k+1} C^i_k \overline{P}^i_{k+1|k} \quad (40)$$

where $\widehat{A}_z$ is the expected value of Θ_z and $\overline{K}_{k+1} = (\Phi \overline{P}_{k+1|k} C^{i^T}_k + G_k S_k)(C^i_k \overline{P}^i_{k+1|k} C^{i^T}_k + R^i_{e,k})^{-1}$. In the ub-KF, $\mathbf{E}[A_z A_z^T]$ can be computed in advance but computation of λ_{max} is required at each step of the algorithm. However, if the size of Z is large, it is more efficient than the modified KF[1].

Lemma III.3: If for an i-th sensor, $\overline{P}^i_{k|k} \geq P^i_{k|k}$, then $\overline{P}^i_{k+1|k} \geq P^i_{k+1|k}$.

Proof: This is proved in the Appendix. ■

Remark III.2: Using Lemma III.3, Lemma III.2, and the induction hypothesis, the following theorem is obtained. The ub-KF maintains the state error covariance which is lower-bounded by the state error covariance of the modified KF.

Theorem III.3: If for an i-th sensor, the ub-KF starts with an initial covariance $\overline{P}^i_{0|0}$, such that $\overline{P}^i_{0|0} \geq P^i_{0|0}$, then $\overline{P}^i_{k|k} \geq P^i_{k|k}$ for all $k \geq 0$.

3) Convergence : Theorem III.4 shows a simple condition when the state error covariance is unbounded.

Theorem III.4: If $(\mathbf{E}[A_z]^T$, $\mathbf{E}[A_z]^T C^{i^T}_k)$ is not stabilizable, or equivalently, $(\mathbf{E}[A_z], C^i_k \mathbf{E}[A_z])$ is not detectable, then there exists an initial covariance $P^i_{0|0}$ such that $P^i_{k|k}$ diverges as $k \to \infty$.

Proof: This is proved in the Appendix. ■

[1] It should be noted that the computation of λ_{max} requires a polynomial number of operations in N while the size of Z can be exponential in N.

IV. Bayesian-based FB KF Fusion with Incomplete Prior Information

In practice, prior information of some but not all the components of $\bar{\alpha}$ are available. For example, when tracking the positioning of a vehicle it is easy to determine the prior position vector of the vehicle (it must be within a certain position range) with certain covariance, but not the velocity of the vehicle, i.e. at what speed it is traveling. Such an incomplete prior problem is addressed in this section using Bayesian-based FB KF. Theorem IV.1 presents the Bayesian-based FB KF with incomplete prior information for an i-th sensor:

Theorem IV.1:

$$Forward\, Run\colon For\; (k=0;\; k<K;\; +k)$$

$$\widehat{\alpha}^i_{k|k} = VK_kV^T\bar{\alpha}^i_{k|k} + VK_k[\Upsilon^i_k - \bar{\nu}] \tag{41}$$

$$\widehat{\alpha}^i_{k+1|k} = VK_kV^T\widehat{\alpha}^i_{k+1|k} + VK_k\Upsilon^i_k - VK_kV^T \tag{42}$$

$$\widehat{P}^i_{k|k} = K_kC^i_kP^i_{k|k-1} \tag{43}$$

$$K_k = C^{i^+}_k[I - P^i_{k|k-1}((I - C^i_kC^{i^T}_k)(P^i_{k|k-1})(I - C^i_kC^{i^T}_k))^+] \tag{44}$$

$$\tilde{K} = K + B^T(I - C^i_kC^{i^T}_k) \tag{45}$$

$$P^i_{k+1|k} = G^i_kQ^i_kG^{i^T}_k - K_kR^i_{e,k}K^T_k \tag{46}$$

$$Backward\, Run\colon For\; (k=0;\; k<K;\; +k)$$

$$\widehat{J}_{k-1|K} = \widehat{P}^i_{k-1|K}\Phi^T_k(\widehat{P}^i_{k-1|K})^{-1} \tag{47}$$

$$\widehat{\alpha}^i_{k-1|K} = \widehat{\alpha}^i_{k-1|k-1} + \widehat{J}_{k-1}(\widehat{\alpha}^i_{k-1|K} - \widehat{\alpha}^i_{k-1|k}) \tag{48}$$

$$\widehat{P}^i_{k-1|K} = \widehat{P}^i_{k-1|k-1} + \widehat{J}_{k-1}(\widehat{J}_{k-1|K} - \widehat{P}^i_{k-1|k})J^T_{k-1} \tag{49}$$

where B_k is any matric of compatible dimensions satisfying $[(P^i_{k|k-1})^{\frac{1}{2}}]^T(I - C^i_kC^{i^+}_k)B_k = 0$, $(P^i_{k|k-1})^{\frac{1}{2}}$ is any square root matrix of $P^i_{k|k-1}$. The optimal gain matrix $\tilde{K}$ is given uniquely by:

$$\tilde{K} = K = C^{i^+}_k[I - P^i_{k|k-1}(I - C^i_kC^{i^+}_k)^{\frac{1}{2}}((I - C^i_kC^{i^+}_k)^{\frac{1}{2}^T}$$

$$P^i_{k|k-1}(I - C^i_k C^{i+}_k)^{\frac{1}{2}})^{-1}(I - C^i_k C^{i+}_k)^{\frac{1}{2}^T}] \tag{50}$$

if and only *if* $[C^i_k, (P^i_{k|k-1})^{\frac{1}{2}}]$ has full row rank, where $(I - C^i_k C^{i+}_k)^{\frac{1}{2}}$ is a full-rank square root of T. Note that variables are derived according with condition of C_k as full row rank.

Proof: This is proved in the Appendix. ■

A. Modified KF With Incomplete Prior Information

In this section, the case with incomplete prior information is outlined. The modification of the KF is focused towards the prediction covariance computing of that case. The prediction covariance when dealing with incomplete prior information is (See Eq. (51)):

$$\begin{aligned} P^i_{k+1|k} &= \mathbf{E}[e^i_{k+1|k} e^{iT}_{k+1|k} | \Upsilon^i_k] \\ &= G_k Q_k G^T_k - K_p R^i_{e,k} K^T_p + \sum_{z \in Z} p_z \Theta_z \widehat{\alpha}^i_{k|k} \widehat{\alpha}^{iT}_{k|k} (\Theta_z - \widehat{\Theta})^T \end{aligned} \tag{51}$$

And here also, given $\widehat{\alpha}^i_{k+1|k}$ and $P^i_{k+1|k}$, $\widehat{\alpha}^i_{k+1|k+1}$ and $P^i_{k+1|k+1}$ are computed as in the standard KF (See Eq. (52–53)).

$$\widehat{\alpha}^i_{k+1|k+1} = K_{k+1}[\Upsilon^i_{k+1} - \bar{\nu}_k] \tag{52}$$

$$P^i_{k+1|k+1} = K_{k+1} C^i_k P^i_{k+1} \tag{53}$$

where $K_{k+1} = \tilde{C}^{i+}_k [I - \tilde{P}^i_{k+1|k}(I - \tilde{C}^i_k \tilde{C}^{iT}_k)(P^i_{k+1|k})]$.

B. Approximating the KF for Incomplete Prior Information

Likewise in Section B, since the computation of $P^i_{k+1|k}$ is the only time-consuming process, two filtering method is proposed to bound $P^i_{k+1|k}$. The same notations have been followed as in Section B.

1) lb-KF: Incomplete Prior Information Case : The lower-bound KF (lb-KF) is the same as the modified KF described in Section IV.A, except $P^i_{k+1|k}$ is approximated by $\underline{P}^i_{k+1|k}$, and $P^i_{k|k}$ by

$\underline{P}^i_{k|k}$. The covariances are updated as:

$$\underline{P}^i_{k+1|k} = G_k Q_k G_k^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k^T \tag{54}$$

$$\underline{P}^i_{k+1|k+1} = V \underline{K}_{k+1} C^i_k \underline{P}^{iT}_{k+1|k} V^T \tag{55}$$

where $\underline{K}_{k+1} = \tilde{C}^{i+}_k [I - \underline{\tilde{P}}^i_{k+1|k} (I - \tilde{C}^i_k \tilde{C}^{iT}_k)(\underline{\tilde{P}}^i_{k+1|k})$.

Lemma IV.1: If $\underline{P}^i_{k|k} \preceq P^i_{k|k}$, then $\underline{P}^i_{k+1|k} \preceq P^i_{k+1|k}$.

Proof: This is proved in the Appendix. ■

2) ub-KF: Incomplete Prior Information Case : Similar to the lb-KF, for an i-th sensor, the upper-bound KF (ub-KF) approximates $P^i_{k+1|k}$ by $\overline{P}^i_{k+1|k}$ and $P^i_{k|k}$ by $\overline{P}^i_{k|k}$. Let $\lambda_{max} = \lambda_{max}(\overline{P}^i_{k|k}) + \lambda_{max}(\widehat{\alpha}^i_{k|k} \widehat{\alpha}^{iT}_{k|k})$, where $\lambda_{max}(S)$ denotes the maximum eigenvalue of S. The covariances are updated as following:

$$\overline{P}^i_{k+1|k} = \lambda_{max} \mathbf{E}[A_z A_z^T] + \overline{K}_k \overline{R}^i_{e,k} \overline{K}^T_k \tag{56}$$

$$\overline{P}^i_{k+1|k+1} = \overline{K}_{k+1} C^i_k \overline{P}^i_{k+1|k} \tag{57}$$

where $\overline{K}^i_{k+1} = \tilde{C}^{i+}_k [I - \overline{\tilde{P}}^i_{k+1|k} (I - \tilde{C}^i_k \tilde{C}^{iT}_k)(\overline{\tilde{P}}^i_{k+1|k})$. In the ub-KF, $\mathbf{E}[A_z A_z^T]$ can be computed in advance but computation of λ_{max} is needed at each step of the algorithm.

Lemma IV.2: If $\overline{P}^i_{k|k} \geq P^i_{k|k}$, then $\overline{P}^i_{k+1|k} \geq P^i_{k+1|k}$.

Proof: This is proved in the Appendix. ■

Using Lemma IV.2, Lemma III.2, and the induction hypothesis, the following theorem is obtained. The ub-KF maintains the state error covariance which is lower-bounded by the state error covariance of the modified KF.

Theorem IV.2: If the ub-KF starts with an initial covariance $\overline{P}^i_{0|0}$, such that $\overline{P}^i_{0|0} \geq P^i_{0|0}$, then $\overline{P}^i_{k|k} \geq P^i_{k|k}$ for all $k \geq 0$.

3) Convergence : The convergence will same as followed in Theorem III.4.

V. Distributed Filtering Fusion

Define n-dimensional master observation variables as:

$$I_{\text{master,k}} = C^T_{\text{master},k} R^{-1}_{\text{master},k} \Upsilon_{\text{master},k},$$

$$I_{\text{master}} = C^T_{\text{master},k} R^{-1}_{\text{master},k} C_{\text{master},k} \tag{58}$$

and n-dimensional local observation variables at sensor i as:

$$I_{i,k} = C^{i^T}_k R^{i^{-1}}_k \Upsilon^i_k,\ I_i = C^{i^T}_k R^{i^{-1}}_k C^i_k \tag{59}$$

where I stands for information matrix. When the observations are distributed among the sensors, see Eq. (2), the master information filter can be implemented by collecting all sensor observations at a central location, or with observation fusion. This is achieved by realizing that master observation variables in (58) as [27]:

$$I_{\text{master},k} = \sum_{i=1}^{l} I_{i,k},\ k \geq 0,\ I_{\text{master}} = \sum_{i=1}^{l} I_i \tag{60}$$

Considering the same domain and ignoring the risk of introducing additional process errors during the domain transformation, let $P_{\text{master},k|k}$ be the updated $a-posteriori$ estimate covariance matrix and $P_{\text{master},k|k-1}$ be the predicted $a-priori$ estimate covariance matrix collected from the master filter at k-th time instant. Also, $P_{\text{master},0|0}$ is the initial error covariance for the master filter. Then the master filtering measurement updates can be given by this alternate information form (See proof [28]):

$$P^{-1}_{\text{master},k|k} \widehat{\alpha}_{\text{master},k|k} = P^{-1}_{\text{master},k|k-1} \widehat{\alpha}_{\text{master},k|k-1} + C^T_{\text{master}} R^{-1}_{\text{master}} \Upsilon_{\text{master},k}$$

$$P^{-1}_{\text{master},k|k} = P^{-1}_{\text{master},k|k-1} + C^T_{\text{master}} R^{-1}_{\text{master}} C_{\text{master}} \tag{61}$$

1) Convergence of the Distributed fusion : The master error covariance matrix and the estimate are given in terms of the local covariances and estimates by

$$P^{-1}_{\text{master},k|k} = P^{-1}_{\text{master},k|k-1} + \sum_{i=1}^{l} (P^{i^{-1}}_{k|k-1} - P^{i^{-1}}_{k|k})$$

$$P^{-1}_{\text{master},k|k}\widehat{\alpha}_{\text{master},k|k} = P^{-1}_{\text{master},k|k-1}\widehat{\alpha}_{\text{master},k|k-1} + \sum_{i=1}^{N}(P^{i^{-1}}_{k|k}\widehat{\alpha}^{i}_{k|k} - P^{i^{-1}}_{k|k-1}\widehat{\alpha}^{i}_{k|k-1}) \tag{62}$$

Proof: Proof follows by noting that the master estimate is given by:

$$\begin{aligned} P^{-1}_{\text{master},k|k}\widehat{\alpha}_{\text{master},k|k} &= P^{-1}_{\text{master},k|k-1}\widehat{\alpha}_{\text{master},k|k-1} + C^{T}_{\text{master}}R^{-1}_{\text{master}}\Upsilon_{\text{master},k} \\ P^{-1}_{\text{master},k|k} &= P^{-1}_{\text{master},k|k-1} + C^{T}_{\text{master}}R^{-1}_{\text{master}}C_{\text{master}} \end{aligned} \tag{63}$$

Since R_{master} is block diagonal, the terms $C^{T}_{\text{master}}R^{-1}_{\text{master}}\Upsilon_{\text{master},k}$ and $C^{T}_{\text{master}}R^{-1}_{\text{master}}C_{\text{master}}$ are decomposed into the sums

$$\begin{aligned} C^{T}_{\text{master}}R^{-1}_{\text{master}}\Upsilon_{\text{master},k} &= \sum_{i=1}^{N} C^{i^{T}}R^{i^{-1}}\Upsilon^{i}_{k} \\ C^{T}_{\text{master}}R^{-1}_{\text{master}}C_{\text{master}} &= \sum_{i=1}^{N} C^{i^{T}}R^{i^{-1}}C^{i} \end{aligned} \tag{64}$$

Noting for the i-th sensor, the estimate and the error covariance are given by

$$\begin{aligned} P^{i^{-1}}_{k|k}\widehat{\alpha}^{i}_{k|k} &= P^{i^{-1}}_{k|k-1}\widehat{\alpha}^{i}_{k|k-1} + C^{i^{T}}R^{i^{-1}}\Upsilon^{i}_{k} \\ P^{-1}_{\text{master},k|k} &= P^{-1}_{\text{master},k|k-1} + C^{i^{T}}R^{i^{-1}}C^{i} \end{aligned} \tag{65}$$

■

VI. Evaluation and Testing

The evaluation and testing are conducted using a coupled tank system at Control Systems Laboratory, Systems Engineering Department, King Fahd University of Petroleum and Minerals (KFUPM).

A. Experimental Setup and Process Data Collection

The data for the bench-marked laboratory-scale two-tank process control system was collected at a sampling rate of 50 milliseconds. Process data was generated through an experimental setup as shown in Fig. 2. The prime objective of the bench-marked dual-tank system is to reach a reference height of 200 ml of the second tank. During this process, several faults were generated such as the leakage faults, sensor faults and actuator faults. Leakage faults were introduced through the pipe clogs of the system, knobs between the first and the second tank, drainage knobs etc. Sensor faults were applied by introducing a gain in the circuit to imitate a fault in the level sensor of the tank. Actuator faults were also evaluated using the motor and pump. A Proportional and Integral (PI) controller was applied in a closed loop configuration to reach the desired height of the second tank. Due to the inclusion of faults, the controller was finding it difficult to reach the desired level. For this reason, the power of the motor was increased from a scale of 0 to 5 volts to a scale of 5 to 18 volts in order to provide it with the maximum throttle to reach the desired level. In doing so, the actuator performed well in achieving its desired level, but it also suppressed the faults of the system. This made the task of detecting the faults even more difficult. After the collection of data, techniques such as settling time, steady state value, and coherence spectra can be used to give an insight of the fault.

In this paper, leakage fault was considered. Hydraulic height and liquid output flow-rate of the second tank are the inputs while leakage fault level on a discrete scale of 1 to 4 was the considered output. Data was collected by introducing leakage fault in the closed loop system.

B. Model of the Coupled Tank System

The two tanks of the coupled tank system are joined together by a network of pipe. The leakage was simulated in the tank by opening the drain valve. A DC motor-driven pump supplied the fluid to the first tank and a PI controller was used to control the fluid level in the second tank by maintaining the level to a specific threshold as shown in Fig. 3.

For the model of the coupled tank system [29], a step input was applied to the DC motor-pump system to fill the first tank. The opening of the drainage valve generated a leakage in the

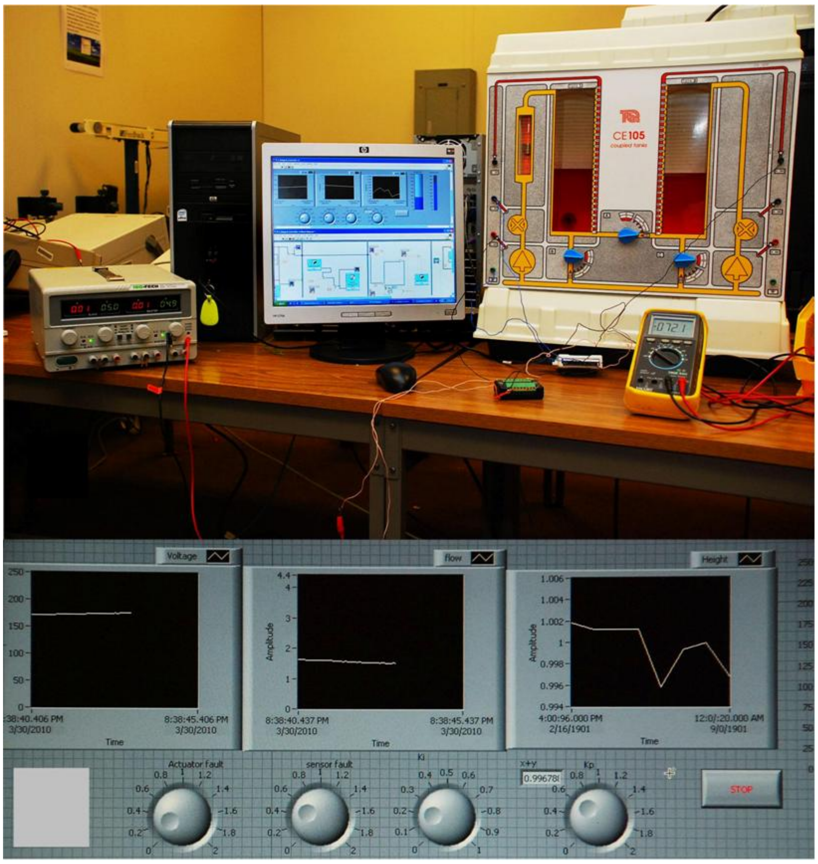

Fig. 2. A – The two tank system interfaced with the LABVIEW through a DAQ and the amplifier for the magnified voltage , B – The LABVIEW setup of the apparatus including the circuit window and the block diagram of the experiment.

tank. Liquid height in the second tank H_2, and the inflow rate Q_i of various type of leakage faults were measured. The National Instruments Labview package was employed to collect these data.

The cascade connection of a DC motor and a pump relating the input to the motor *u* and the flow Q_i can be expressed as a first-order system:

$$\dot{Q}_i = -a_m Q_i + b_m \phi_u \tag{66}$$

where a_m and b_m are the parameters of the motor-pump system, ϕ_u is a dead-band and saturation type of nonlinearity and $\dot{Q}_i$ is the rate of change of input flow. It is assumed that the leakage Q_ℓ

occurs in tank 1 and is given by:

$$Q_\ell = C_{da}\sqrt{2gH_1} \tag{67}$$

where C_{da} is the discharge coefficient of the leakage valve in tank 1, H_1 is the liquid height in the first tank and $g = 980\ cm/\sec^2$ is the gravitational constant. With the inclusion of the leakage, the liquid level system is modeled by:

$$A_1\dot{H}_1 = Q_i - C_{db}\varphi(H_1 - H_2) - C_{da}\varphi(H_1) \tag{68}$$

$$A_2\dot{H}_2 = C_{db}\varphi(H_1 - H_2) - C_{dc}\varphi(H_2) \tag{69}$$

where $\varphi(.) = sign(.)\sqrt{2g(.)}$, $Q_\ell = C_{da}\varphi(H_1)$ is the leakage flow rate, $Q_0 = C_{dc}\varphi(H_2)$ is the output flow rate, A_1 and A_2 are the cross-sectional areas of the two tanks, C_{db} and C_{dc} are the discharge coefficient of the leakage valve in tank 2 and output valves respectively.

The model of the two-tank fluid control system, shown in Fig. 3, is of a second order and is nonlinear with a smooth square-root type of nonlinearity. For design purposes, a linearized model of the fluid system is required and is given below in (70–71):

$$\dot{h}_1 = b_1 q_i - (a_1 + \gamma)h_1 + a_1 h_2 \tag{70}$$

$$\dot{h}_2 = a_2 h_1 - (a_2 - \beta)h_2 \tag{71}$$

where h_1 and h_2 are the increments in the nominal (leakage-free) values to heights H_1^0 and H_2^0. Parameters γ and β indicate the amount of leakage and output flow rate respectively, where $\gamma = \frac{C_{da}}{2\sqrt{2gH_1^0}}$ and $\beta = \frac{C_{dc}}{2\sqrt{2gH_2^0}}$. Also $b_1 = \frac{1}{A_1}$, $a_1 = \frac{C_{db}}{2\sqrt{2g(H_1^0 - H_2^0)}}$ and $a_2 = a_1 + \frac{C_{dc}}{2\sqrt{2gH_2^0}}$.

A PI controller, with gains k_p and k_I, was used to maintain the level of the tank 2 at the desired reference input r as:

$$\begin{aligned} \dot{x}_3 &= e = r - h_2 \\ u &= k_p e + k_I x_3 \end{aligned} \tag{72}$$

where $\dot{x}_3$ is the rate of change of error, r is the reference height of tank 2 .i.e. 200 ml and h_2 is the height of the tank 2 achieved and u is the control input. The linearized model of the entire system formed by the motor, pump, and the tanks is given by:

$$\dot{\alpha} = \Phi\alpha + \Gamma r \quad \Upsilon = C\alpha \tag{73}$$

where

$$\alpha = \begin{bmatrix} h_1 \\ h_2 \\ x_3 \\ q_i \end{bmatrix}, \ \Phi = \begin{bmatrix} -a_1-\gamma & a_1 & 0 & b_1 \\ a_2 & -a_2-\beta & 0 & 0 \\ -1 & 0 & 0 & 0 \\ -b_m k_p & 0 & b_m k_I & -a_m \end{bmatrix}, \tag{74}$$

$$\Gamma = \begin{bmatrix} 0 & 0 & 1 & b_m k_p \end{bmatrix}^T, \ C = [1 \quad 0 \quad 0 \quad 0]$$

Here q_i, q_ℓ, q_0, h_1 and h_2 are the increments in Q_i, Q_ℓ, Q_0, H_1^0 and H_2^0 respectively. The parameters a_1 and a_2 are associated with linearization of the liquid level model. Variables γ and β are the leakage and output flow rate, respectively. Thus, $q_\ell = \gamma h_1$ and $q_0 = \beta h_2$.

Remark VI.1: During the implementation process, $sign(.)$ can be approximated with $arc\,tangent$. A relationship for approximation can be expressed as follows:

$$sign(x) = \arctan(\frac{\alpha}{\sqrt{1-\alpha^2}}), \ where\ \alpha < 1 \tag{75}$$

C. Evaluation Results

Experimental results of the proposed distributed approximate estimation with two cases of prior knowledge are presented in this section. The experiment were performed on the coupled tank system [29]. Firstly, the data collected from the plant was initialized and the parameters were later optimized. This comprised of the pre-processing and normalization of the data. Secondly, a networked control system with wired communication was developed in a Matlab environment as shown in the Fig. 1. In the simulation, the performance of the modified KF algorithms developed

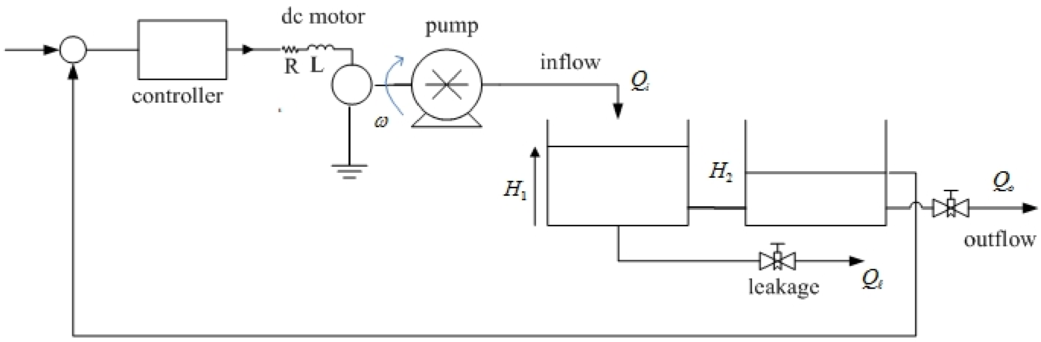

Fig. 3. Process control system: A lab-scale two-tank system

was studied for types of prior information against the standard Bayesian-based Kalman smoother, which assumed no communication errors. Subsequently, results demonstrating the effectiveness of the lb-KF and ub-KF are presented. For each test case, the modified Bayesian-based KF was compared with the standard Bayesian-KF. The comparisons were shown for various cases. Moreover, the time computation of state estimates was made and the results were shown in Table I and II respectively.

Remark VI.2: It should be noted here that height sensor in coupled tank setup has been interfacted with Labview for the purpose of data fusion as shown in Fig. 2 Moreover, the potency of the leakage fault i.e small, medium or large is being defined with the help of the leakage knobs facility between the two tank tanks and drainage as shown in the main diagram of Fig. 2.

1) Leakage Fault: Estimates and Covariance Comparison with Complete and Incomplete prior Information Cases

The Bayesian-based FB KF was evaluated to address the leakage fault of the plant. Simulations were made for the α-estimate and the covariance of each case. Comparisons of various levels of leakage i.e. no, small, and medium intensity of leakage faults, and distributed estimation results are shown in Fig. (5–12).

For complete prior information situation, it is observed that the covariance (see Fig. 4) and the estimate (see Fig.5) of the distributed structure is clearly performing well as compared to the other profiles. Similar performance is seen for the covariance and estimate of modified filter implementation with ub (see Fig. 6 for covariance of ub scheme and see Fig.7 for estimate of ub scheme) and lb (see Fig.8 for covariance of lb scheme and see Fig.9 for estimate of lb scheme). The advantage of using the modified upper and lb filters is distinctly illustrated in the time computation comparison and mean square error (MSE) as discussed in the next Section.

For incomplete prior information scenario, it can be seen for the estimate profile (see Fig. 10) that the distributed structure is clearly performing well as compared to the other profiles. In addition, the covariance and estimate of the modified filter implementation with ub (see Fig. 11) are performing equally well for distributed structure. Other estimates shown in Fig. 12 also elaborate the performance of distributed estimation and estimation of the modified filters. [2]

D. Time Computation and MSE

In this section, the time computation and MSE of different methods is discussed. They were employed for calculating the estimates and covariances of the state with complete prior and incomplete prior information. An equal number of 5 iterations were performed for achieving each and every of the estimate.

For the case of complete prior information (See Table I), the iteration time of the basic Bayesian-based FB KF is taking the maximum number of time for the computation despite of its optimal structure as compared to the regular KF. On the other hand, the modified versions of ub and lb filters are performing well in time computation for the leakage fault. More precisely, the performance of distributed version and lb and ub are shown in Fig. 13-15, where the performance of distributed version and modified filters is quiet visible.

[2] *Fig. 4–12 shows the comparison of estimates and covariance for types of $a - priori$ information cases. In all these figures x-axis shows the number of observations taken at a sampling rate of 50 milliseconds of time, and y-axis shows the α-estimate which presents the estimate of a particular state.*

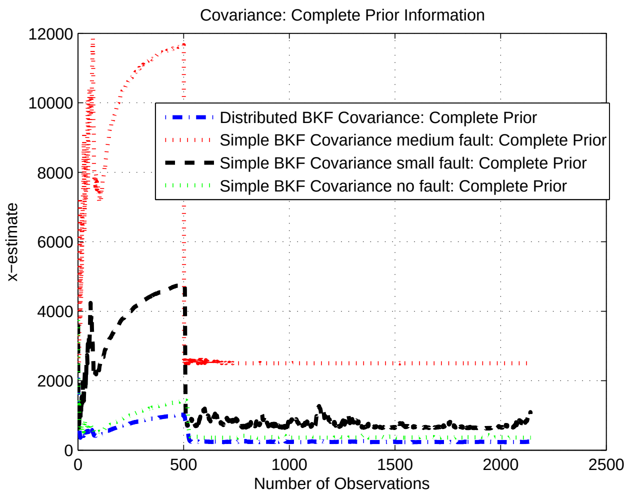

Fig. 4. Comparison of covariance for complete prior information for leakage fault

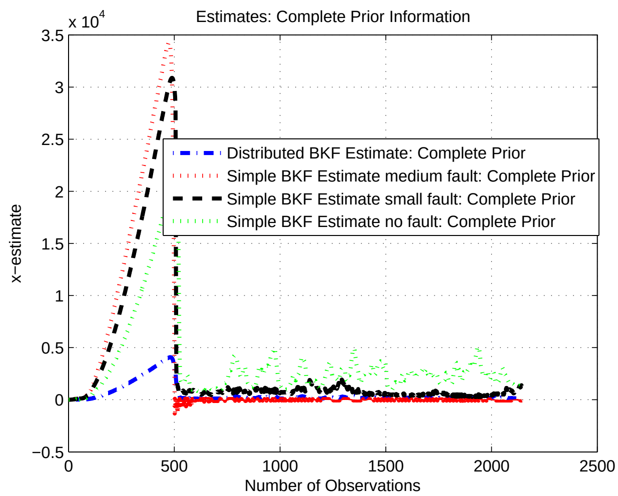

Fig. 5. Comparison of estimates for complete prior information for leakage fault

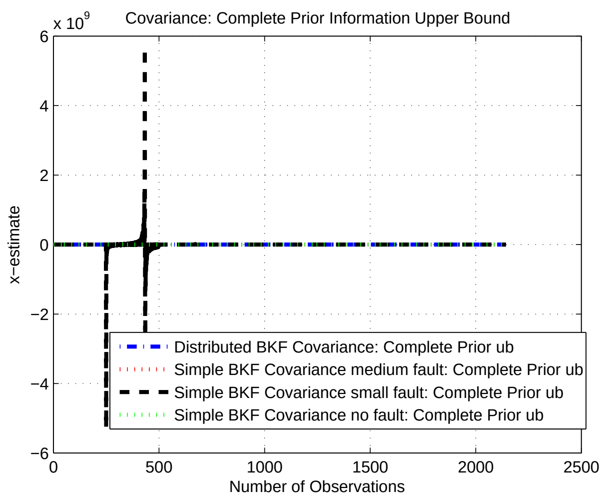

Fig. 6. Comparison of covariance for complete prior information for leakage fault with ub modified filter

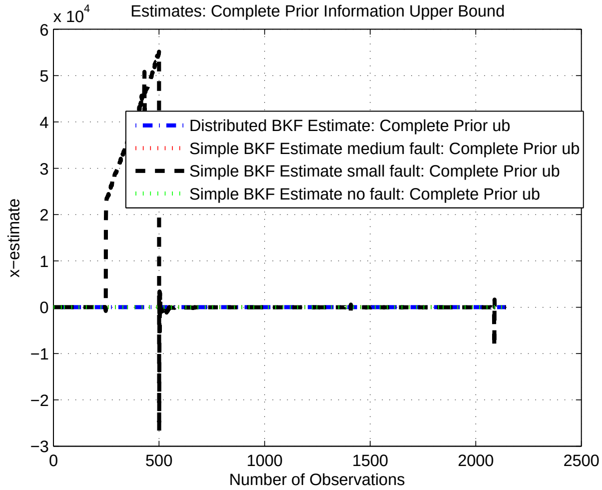

Fig. 7. Comparison of estimates for complete prior information for leakage fault with ub modified filter

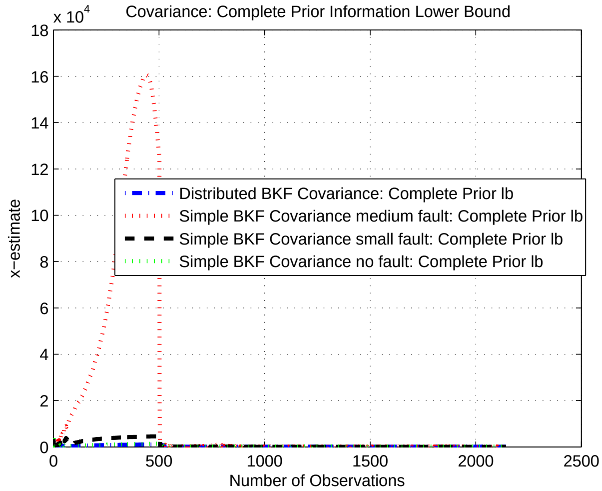

Fig. 8. Comparison of covariance for complete prior information for leakage fault with lb modified filter

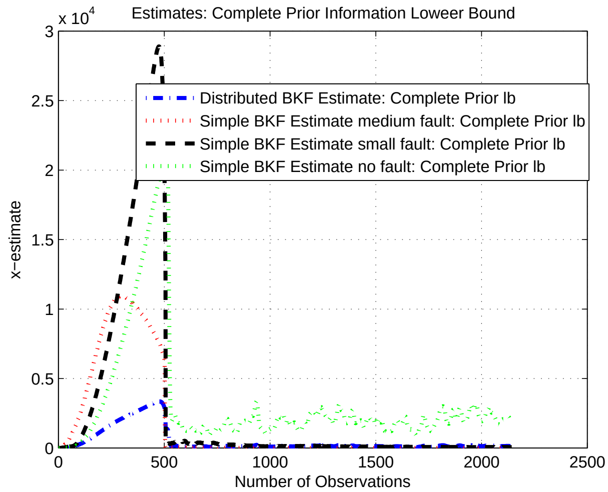

Fig. 9. Comparison of estimates for complete prior information for leakage fault with lb modified filter

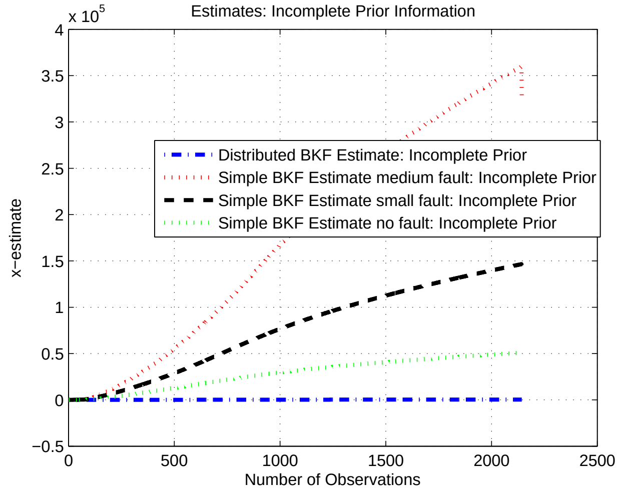

Fig. 10. Comparison of estimates for incomplete prior information for leakage fault

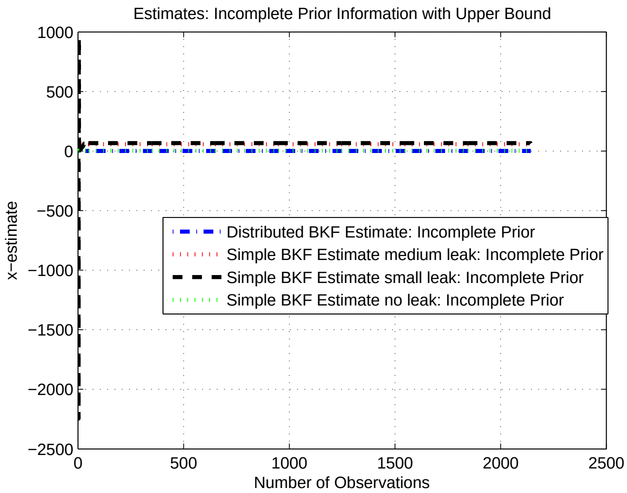

Fig. 11. Comparison of estimates for incomplete prior information for leakage fault

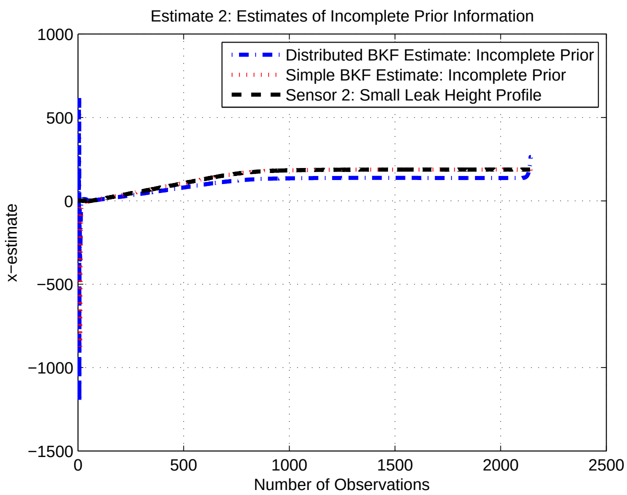

Fig. 12. Estimate 2: Comparison of estimates for complete prior information

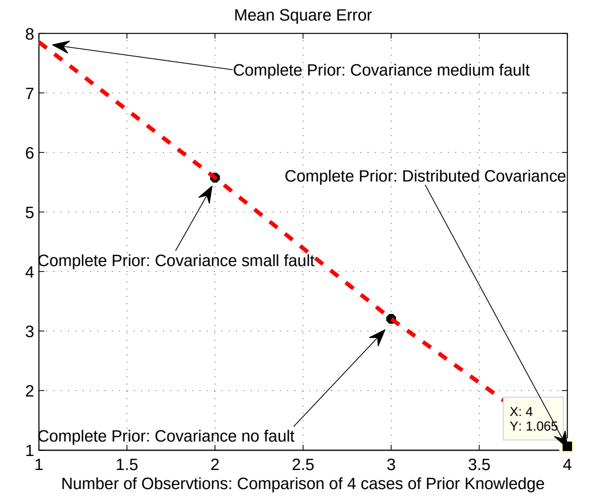

Fig. 13. MSE for complete prior case

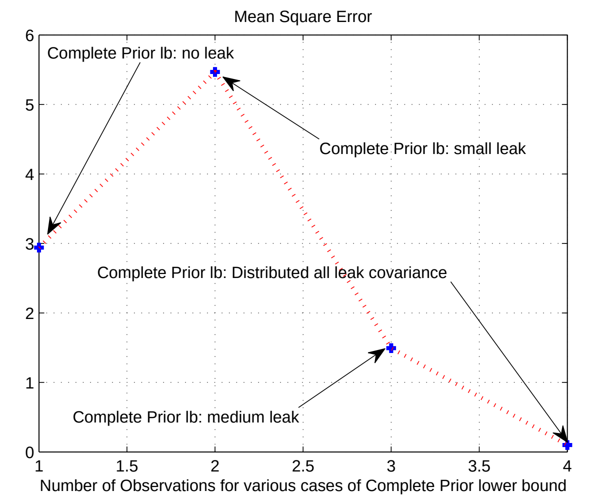

Fig. 14. MSE for complete prior case with lb

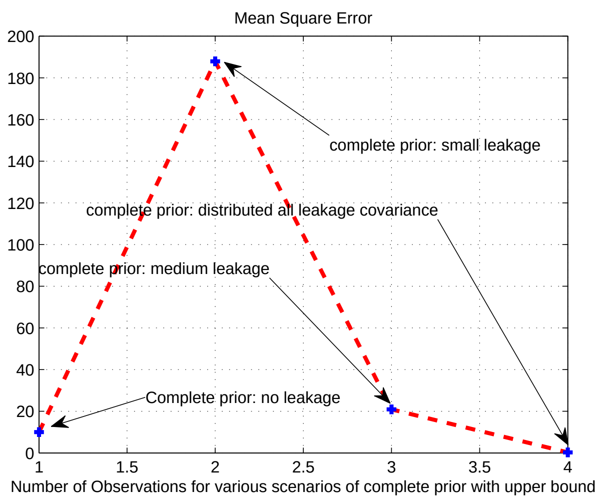

Fig. 15. MSE for complete prior case with ub

TABLE I

CASE I: TIME COMPUTATION COMPARISON FOR COMPLETE PRIOR INFORMATION

Types of Filters	Bayesian FB KF	Bayesian FB KF with ub	Bayesian FB KF with lb
Time computation (sec)	**15.707692**	**12.656498**	**12.993867**

TABLE II

CASE II: TIME COMPUTATION COMPARISON FOR INCOMPLETE PRIOR INFORMATION

Types of Filters	Bayesian FB KF	Bayesian FB KF with ub	Bayesian FB KF with lb
Time computation (sec)	**13.451193**	**12.996375**	**11.915037**

For the case of incomplete prior information (See Table II) the basic Bayesian-based FB KF is taking comparatively more time as compared to the modified versions of lb and ub filters. The performance of the modified filters was consistent even with a leakage fault.

VII. CONCLUSIONS

In this paper, a distributed approximate estimation scheme was proposed. It was being devised using Bayesian-based FB KF for a singular stochastic linear system. Two test cases were considered for approximate estimation, with 1) complete $a-priori$ information 2) incomplete $a-priori$ information. The proposed scheme was able to minimize the time complexity with conditions of showing dependency on $a-priori$ knowledge. Moreover, the performance was enhanced by using a distributed filtering architecture. The proposed scheme was evaluated on a coupled tank system using various fault scenarios. It ensured the effectiveness of the approach under different $a-priori$ knowledge cases.

APPENDIX

A. *Proof of Theorem III.1*

For an i-th sensor linear estimation of α_k^i using data Υ_k^i with linear model $\Upsilon_k^i = C_k^i \alpha_k^i + \nu_k$, the prior information consists of $\bar{\alpha}_k^i$ and $\bar{\nu}_k$, and $C_{\alpha_k}^i = cov(\alpha_k^i)$, $C_{\nu_k} = cov(\nu_k)$, and $C_{\alpha_k^i \nu_k} = cov(\alpha_k^i, \nu_k)$. It should be noted that the prior information mean the $a-priori$ information about

α_k^i, that is $\bar{\alpha}_k^i$, $C_{\alpha_k}^i$, and C_{α_k,ν_k}^i.

For the dynamic case, as in KF,

$$\begin{aligned}
\widehat{\alpha}_{k|k}^i &= \mathbf{E}[\alpha_k^i|\Upsilon_k^i]^T = [\bar{\alpha}_k^i|\Upsilon_k^i] \\
&= \bar{\alpha}_k^i + C_{\alpha_k\Upsilon_k}^i C_{\Upsilon_k^i}^{i+}(\Upsilon_k^i - \bar{\Upsilon}_k^i),\ \bar{\alpha}_k^i = \mathbf{E}[\alpha_k^i] \\
P_{k|k}^i &= \text{MSE}(\widehat{\alpha}_{k/k}^i) = \mathbf{E}[(\alpha_k^i - \widehat{\alpha}_{k|k}^i)(\alpha_k^i - \widehat{\alpha}_{k|k}^i)^T] \\
&= C_{\alpha_k^i} - C_{\alpha_k^i\Upsilon_k^i} C_{\Upsilon_k^i}^{+} C_{\alpha_k^i\Upsilon_k^i}^{T}
\end{aligned}$$

where $C_{\Upsilon_k}^{i+}$ is the Moore-Penrose pseudo-inverse of $C_{\Upsilon_k}^i$, which equals $C_{\Upsilon_k}^{-1}$ whenever $C_{\Upsilon_k}^{-1}$ exists. With few exceptions, however, it is unrealistic since its computational burden increases rapidly with time (method for decreasing time computation complexity is applied in the next section using modified KF functions of ub and lb.

$$\begin{aligned}
\widehat{\alpha}_{k|k}^i &= \mathbf{E}[\alpha_k^i|\Upsilon_k^i]^T = \mathbf{E}[\alpha_k^i|\Upsilon_k^i, \Upsilon_{k-1}^i]^T = \widehat{\alpha}_{k|k-1}^i + K_k\bar{\Upsilon}_{k|k-1}^i \\
P_{k|k}^i &= \text{MSE}(\widehat{\alpha}_{k|k}^i) = \text{MSE}(\widehat{\alpha}_{k|k-1}^i) - K_k C_k^i \bar{\Upsilon}_{k|k-1}^i K_k^T
\end{aligned}$$

where $\tilde{\alpha}_{k|k-1}^i = \alpha_k^i$ - $\widehat{\alpha}_{k|k-1}^i$, $K_k = C_{\tilde{\alpha}_{k|k-1}^i \tilde{\Upsilon}_{k|k-1}^i} C_{\tilde{\alpha}_{k|k-1}^i}^{i+}$, $\tilde{\Upsilon}_{k|k-1}^i = \Upsilon_k^i$- $\mathbf{E}[\Upsilon_k^i|\Upsilon_{k-1}^i]^T$.

Let $A = P_{k|k}^i$ and $\Phi_k^i = \zeta$. Equation (31) follows from the following:

$$\begin{aligned}
&(\zeta P_{k|k}^i C_k^{i^T} + A)(C_k^i + C_k^i A)^{-1} \\
&= \{\zeta[C_{\alpha_k}^i - (C_{\alpha_k}^i C_k^{i^T} + A)(C_k^i C_{\alpha_k}^i C_k^{i^T} + C_k^i + C_k^i A + (C_k^i A)^T)^{-1} \\
&\ .\ (C_x C_k^{i^T} + A)^T] C_k^{i^T} + A\}(C_k^i + C_{k^i} A)^{-1} \\
&= (\zeta C_\alpha + C_k^{i^T} + A)[I - (C_k^i C_x C_k^{i^T} + C_k^i + C_k^i A + (C_k^i A)^T)^{-1} \\
&\ .\ (C_k^i C_\alpha C_k^{i^T} + (C_k^i A)^T)](C_k^i + C_k^i A)^{-1} \\
&= (\zeta C_\alpha C_k^{i^T} + A)(C_k^i C_\alpha C_k^{i^T} + C_k^i + C_k^i A + (C_k^i A)^T)^{-1} \\
&\ .\ (C_k^i + C_k A)(C + C_k A)^{-1} \\
&= (\zeta C_\alpha C_k^{i^T} + A)(C_\Upsilon + C_k^i A)^{-1}
\end{aligned}$$

B. Proof of lemma III.1

Using (34), the following can be achieved

$$\begin{aligned} P^i_{k+1|k} - \underline{P}^i_{k+1|k} &= \mathbf{E}[A_z P^i_{k|k} A_z^T] + \mathbf{E}[A_z \widehat{\alpha}^i_{k|k} \widehat{\alpha}^T_{k|k} A_z^T] \\ &- \widehat{\Theta} \widehat{\alpha}^i_{k|k} \widehat{\alpha}^T_{k|k} \widehat{A}_z^T - \widehat{A}_z \underline{P}^i_{k|k} \widehat{A}_z^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k + K_k R^i_{e,k} K_k \\ &= P_1 + P_2 \end{aligned} \tag{76}$$

where $P_1 = \mathbf{E}[A_z P^i_{k|k} A_z^T] - \widehat{A}_z \underline{P}^i_{k|k} \widehat{A}_z^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k$ and $P_2 = \mathbf{E}[A_z \widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k} A_z^T] - \widehat{A}_z \widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k} \widehat{A}_z^T + K_k R^i_{e,k} K_k$.

If $P_1 \geq 0$ and $P_2 \geq 0$, then $P^i_{k+1|k} - \underline{P}^i_{k+1|k} \geq 0$

$$\begin{aligned} P_1 &= \mathbf{E}[A_z P^i_{k|k} A_z^T] - \widehat{A}_z \underline{P}^i_{k|k} \widehat{A}_z^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k^T - \widehat{A}_z P^i_{k|k} \widehat{A}_z^T + \widehat{A}_z P^i_{k|k} \widehat{A}_z^T \\ &= \mathbf{E}[A_z P^i_{k|k} A_z^T] - \widehat{A}_z P^i_{k|k} \widehat{A}_z^T + \widehat{A}_z (P^i_{k|k} - \underline{P}^i_{k|k}) \widehat{A}_z^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k^T \end{aligned} \tag{77}$$

Since $P^i_{k|k}$ is a symmetric matrix, $P^i_{k|k}$ can be decomposed into $P^i_{k|k} = U_1 D_1 U_1^T$, where U_1 is a unitary matrix and D_1 is a diagonal matrix. Hence,

$$\begin{aligned} P_1 &= \mathbf{E}[(A_z U_1 D_1^{1/2})(A_z U_1 D_1^{1/2})^T] - \mathbf{E}[(A_z U_1 D_1^{1/2})] \mathbf{E}[(A_z U_1 D_1^{1/2})]^T \\ &+ \widehat{A}_z (P^i_{k|k} - \underline{P}^i_{k|k}) \widehat{A}_z^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k^T \\ &= Cov[(A_z U_1 D_1^{1/2}] + \widehat{A}_z (P^i_{k|k} - \underline{P}^i_{k|k}) \widehat{A}_z^T - \underline{K}_k \underline{R}^i_{e,k} \underline{K}_k \end{aligned} \tag{78}$$

where $Cov[C^i_k]$ denotes the covariance matrix of C^i_k. Since a covariance matrix is positive definite and $P^i_{k|k} - \underline{P}^i_{k|k} \geq 0$ by assumption, $P_1 \geq 0$. P_2 is a covariance matrix since $\widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k}$ is symmetric, hence $P_2 \geq 0$.

C. Proof of lemma III.2

Here, matrix inversion lemma will be used which says that $(A+UCV)^{-1} = A^{-1} - A^{-1}U(C^{-1} + VA^{-1}U)^{-1}VA^{-1}$ where A, U, C and V all denote matrices of the correct size. Applying the matrix inversion lemma to (36), $P^i_{k+1|k+1} = (P^{i^{-1}}_{k+1|k} + C^{i^T}_k R^{i^{-1}}_{e,k} C^i_k)^{-1}$ is obtained. Let $P = P^i_{k+1|k}$ and

$\underline{P} = \underline{P}^i_{k+1|k}$. Then $P \geq \underline{P} \Rightarrow P^{-1} \leq \underline{P}^{-1}$. Also, $P^{-1} + C_k^{i^T} R_{e,k}^{i^{-1}} C_k^i \leq \underline{P}^{-1} + C_k^{i^T} R_{e,k}^{i^{-1}} C_k^i \Rightarrow (P^{-1} + C_k^{i^T} R_{e,k}^{i^{-1}} C_k^i)^{-1} \geq (\underline{P}^{-1} + C_k^{i^T} R_{e,k}^{i^{-1}} C_k^i)^{-1}$. Thus,

$$P^i_{k+1|k+1} \geq \underline{P}^i_{k+1|k+1} \tag{79}$$

D. *Proof of lemma III.3*

Let $M = \widehat{\alpha}^i_{k|k} \widehat{\alpha}^{i^T}_{k|k}$ and I be an identity matrix. Then using (34), the following is obtained

$$\begin{aligned}
\overline{P}^i_{k|k} - P^i_{k|k} &= \lambda_{max} \mathbf{E}[A_z A_z^T] \\
&- \mathbf{E}[A_z P^i_{k|k} A_z^T] - \mathbf{E}[A_z M A_z^T] \\
&- K_p R^i_{e,k} K_p^T + \overline{K}_p \overline{R}^i_{e,k} \overline{K}_p^T \\
&= \mathbf{E}[A_z (\lambda_{max} (\overline{P}^i_{k|k}) I - P^i_{k|k}) A_z^T] \\
&+ \mathbf{E}[A_z (\lambda_{max}(M) I - M) A_z^T] \\
&- K_p R^i_{e,k} K_p^T + \overline{K}_p \overline{R}^i_{e,k} \overline{K}_p^T
\end{aligned} \tag{80}$$

Since, $\overline{P}^i_{k|k} \geq P^i_{k|k}$ and $\lambda_{max}(S) I - S \geq 0$ for any symmetric matrix S, $\overline{P}^i_{k|k} - P^i_{k|k} \geq 0$.

E. *Proof of theorem III.4*

Let us consider the lb-KF. Let $\underline{P}^i_k = \underline{P}^i_{k|k}$. $\psi = G_k Q_k G_k^T$, $\widehat{A}_z = \mathbf{E}[A_z]$, and $\Phi = -(C_k^i \widehat{A}_z \underline{P}^i_k \widehat{A}_z^T C_k^{i^T} + C_k^i \psi C_k^{i^T} + R^i_{e,k})^{-1} (C_{k^i} \psi + C_k^i \widehat{A}_z \underline{P}^i_k \widehat{\Theta}^T)$.

Then based on Riccati difference equation [30], $\underline{P}_{k+1}$ can be expressed as:

$$\begin{aligned}
\underline{P}^i_{k+1} &= \widehat{A}_z \underline{P}^i_k \widehat{A}_z^T + \psi - \Phi^T (C_k^i \widehat{A}_z \underline{P}^i_k \widehat{A}_z^T C_k^{i^T} + C_k^i \psi C_k^{i^T} + R^i_{e,k}) F \\
&= (\widehat{A}_z^T + \widehat{A}_z^T C_k^T F)^T \underline{P}^i_k (\widehat{A}_z^T + \widehat{A}_z^T C_k^{i^T} \Phi) + \Phi^T (C_k^i \psi C_k^{i^T} + R^i_{e,k}) \Phi + \psi C_k^{i^T} \Phi \\
&+ \Phi^T C_k^i \psi + \psi
\end{aligned} \tag{81}$$

Hence, if $(\widehat{A}_z^T + \widehat{A}_z^T C_k^{i^T} \Phi)$ is not a stability matrix, for some $\underline{P}^i_0 \leq P^i_{0|0}$. $\underline{P}^i_{k|k}$ diverges as $k \to \infty$. Since the state error covariance of the lb-KF diverges and $\underline{P}^i_{k|k} \leq P^i_{k|k}$ for all $k \geq 0$ (Theorem

III.2), $P^i_{k|k}$ diverges as $k \to \infty$. Here $P^i_{k|k}$ can be $\Phi_k P^i_{k+1|k} \Phi^T_k - K_k C^i_k P^i_{k+1|k}$ for 'complete' prior case and $K_k C^i_k P^i_{k|k-1}$ for 'incomplete' prior case respectively.

F. Proof of theorem IV.1

By explanation of B_k, the problem can be considered for incomplete prior information with C^i_k and C^i_k replaced by the $\tilde{C}^i_k$ and $\tilde{C}^i_k$ respectively, where, from the proof of Theorem IV.1, the estimate is $u = V^T \alpha_k$, where V is an orthogonal matrix. This means that Theorem IV.1 is applicable now to u. Therefore:

$$\widehat{\alpha} = V\widehat{u},\ P = V\mathbf{MSE}(\widehat{u})V^T$$

The uniqueness result thus follows from Theorem IV.1.

G. Proof of lemma IV.1

Using (51) gets

$$\begin{aligned} P^i_{k+1|k} - \underline{P}^i_{k+1|k} &= \mathbf{E}[A_z \widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k} A^T_z] - K_{p,k} R^i_{e,k} K^T_{p,k} \\ &- \widehat{A}_z \widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k} \widehat{A}^T_z + \underline{K}_{p,k} \underline{R}^i_{e,k} \underline{K}^T_{p,k} \\ &= P_1 + P_2 \end{aligned} \tag{82}$$

where $P_1 = -K_{p,k} R^i_{e,k} K^T_{p,k}$ and $P_2 = \mathbf{E}[A_z \widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k} A^T_z] - \widehat{A}_z \widehat{\alpha}_{k|k} \widehat{\alpha}^T_{k|k} \widehat{A}^T_z - \underline{K}_{p,k} \underline{R}^i_{e,k} \underline{K}^T_{p,k}$.

Since $P^i_{k|k}$ is a symmetric matrix, $P^i_{k|k}$ can be decomposed into $P^i_{k|k} = U_1 D_1 U^T_1$, where U_1 is a unitary matrix and D_1 is a diagonal matrix, but here there is no $P^i_{k|k}$ for P_1.

H. Proof of lemma IV.2

Let $M = \widehat{\alpha}^i_{k|k} \widehat{\alpha}^T_{k|k}$ and I be an identity matrix. Then using (51) gets

$$\overline{P}^i_{k|k} - P^i_{k|k} = \mathbf{E}[A_z(\lambda_{max}(M) I - M) A^T_z] + \widehat{A}_z M \widehat{A}^T_z + \overline{K}_{p,k} \overline{R}^i_{e,k} \overline{K}^T_{p,k}$$

$$- K_{p,k} R^i_{e,k} K^T_{p,k} + G_k Q_k G^T_k \tag{83}$$

Since, $\overline{P}^i_{k|k} \geq P^i_{k|k}$ and $\lambda_{max}(S)I - S \geq 0$ for any symmetric matrix S, $\overline{P}^i_{k|k} - P^i_{k|k} \geq 0$.

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